Interpretation in Game Theory and Existence of a Universal Type Space

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Abstract

The essay investigates the relation between a semantic and a syntactic representation of epistemic characteristics of players in a incomplete information game. It is shown that the two approaches are not always equivalent. We provide sufficient conditions for this equivalence to hold. Moreover, we prove that these conditions are also sufficient for the existence of the so-called universal type space. We identify some of its properties, namely compactness and being self-evident. As a result, the wide-spread use of well-behaved frameworks to model interactive situations is justifiable.

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1 Introduction

There are two main reasons to investigate the role of a language and its interpretation in game theory.

The first one is related to cheap talk games: in this branch of the literature the meaning of the language used for costless messages is subsumed because the attention is on the credibility of the messages. One exception is Farrell (1992), who studied the meaning of cheap talk in a dynamic game with incomplete information.

The second reason is less apparent and originates from the literature on the foundation of game theory in incomplete information settings. Since Harsanyi's seminal paper on games with incomplete information the importance of interactive reasoning and the role of high order uncertainty has become clear. As a matter of fact, beliefs over opponents' action, beliefs over the beliefs of the opponents and so on become relevant. As proposed by Harsanyi (1967-68) hierarchies of beliefs should be implicitly represented through a mathematical structure called a "type space". A type summarizes all payoff properties and the beliefs concerning a certain player and a so-called universal type space consists of all possible types for each player. The existence of a universal type space has been proved in various papers under different assumptions (see [13], Brandenburger and Dekel (1987)[2] and [7]) and its existence justifies the Bayesian framework as a tool for modeling game theoretic contexts. If there is not such space we should stick to the explicit description of agents' reasoning.

Said that, the use of a type space is justified only if it corresponds to the explicit description of the different degree of sophistication of agents. The description, usually, is illustrated by means of a language, namely English but very often a formal language as epistemic logic, then the type space is just one possible interpretation or representation of this explicit description. Hence, the generic problem of interpretation of a language lies at the very basis of the existence and the well-founding of the type space as a mathematical device to model the interactive reasoning of the players.

In the economic literature the issues concerning the existence of universal type space were twofold. On the one hand there are measure theoretic problems usually studied using the semantic approach (refer to [8]). On the other hand there are purely logical ones. In this category, we mention Brandenburger and Keisler (2003), who proved an impossibility theorem for a specific type space, namely a possibility structure.

In the so-called semantic approach the epistemic characteristics of the players are represented by a structure consisting of a set Ω of states of the world, together with a partition Π_i of Ω, for each player i. The element of Π_i, containing an element ω of the set Ω, represents player i knowledge in state ω. This approach is purely set theoretic.
A possible alternative approach is the syntactic approach which sticks to the logic perspective. It is constituted by a set of propositions expressed in a formal language, which is expressive enough to describe the epistemic properties of the players. Logical relations between the various propositions are described by formal rules of inference.

The semantic approach is used more frequently in the economic literature. But, in this framework the complete description of players' reasoning depends crucially on the properties of the external setting[1] and on a informal, but evocative, description of higher order degree of reasoning (see [7] and [9]). Using a semantic approach, the hierarchical construction of a player's type hinges on the explicit description of players' reasoning, but this justification relies on the implicit correspondence of any subset of Ω to a well-formed expression of a non-specified language. Instead, in the syntactic approach this description is explicit and it is built using a formal language, namely epistemic or modal logic. The results are logical deduction from axioms stated in the formal language. By construction, this kind of procedure is absent of any hidden assumption. This fact leads us to start our analysis from a linguistic perspective.

The relation between the semantic and syntactic approaches has been already studied by Aumann ([1]), Fagin et al. ([4]), who analyzed it by means of modal logic (Chellas (1998)), and Kaneko et al. ([10]), who were interested in the broader issue of the characterization of the concept of Common Knowledge in various logics.

In this paper we investigate the relation between the two approaches and we determine

[1] Mertenz and Zamir (1995) [13], for example, assumed that the set of external states is a topological space.

sufficient conditions for the universal type space to exist. As a preliminary step we need to choose the formal language, which could be used by the players for communication and by the researcher to analyze their strategic interaction. In economics the standard choice has been epistemic logic[2]. One reason to consider a different formal language from modal logic is the need to explicitly define arbitrarily high order reasoning and common knowledge without any hidden assumption. Hence, we will use the "first-order logic" which will prove to be a more flexible and natural language. It is suited for modeling the interactive contexts and it has been studied extensively in the logic literature. This choice will be crucial for stating the equivalence between the two approaches that are commonly used in the economic literature for modelling games with incomplete information. For this aim, we use the tools provided by mathematical logic[3]. Specifically, we will employ techniques borrowed from model theory. Model theory is a branch of mathematical logic that considering which mathematical models satisfy a certain theory. We completely separate the two types of analysis (semantic and syntactic), in order to disentangle purely logic issues (see Brandenburger and Keisler (2003)) from purely semantic or set-theoretic ones (see Heifetz and Samet (1998) [8]) and we find conditions, in purely linguistic terms, that exclude both[4].

The issue of the relation between these two approaches and of the interpretation of a language will be connected with the existence of the so-called universal type space. We will determine sufficient conditions for the existence of a universal type space and we will also deduce some of the properties of the so-called universal type space.

The essay is organized as follows. Section 2 introduces the syntactic and semantic approaches and then builds a formal language, which is related with the syntactic approach. In Section 3, allowing only for a finite order of sophisticated reasoning, we prove the equivalence between the semantic and syntactic approach[5]. In Section 4 we prove that this equivalence breaks down if we

[2] Epistemic logic has been widely used in economics for dealing with the rationales of knowledge and beliefs when multiple agents interact.

[3] We mention that the distinction between expession and their intended meanings is the very starting point in logic and not by chance this fact is stated as syntax versus semantics of a language.

[4] As a by-product of our construction, we will be able to state plainly the connection between the individual knowledge operator and the common knowledge operator.

[5] Aumann has already proved this equivalence using a different method.

allow for unbounded length in the description of players' reasoning. In Section 5 we determine the sufficient conditions[6] under which the equivalence is still valid and the universal type space exists. Lastly we are able to identify some characteristics of the universal type space, namely it is compact and it contains at least a self-evident event.

2 Syntactic approach and Semantic approach as $\mathcal{L}$-structures

> "There are some obvious correspondences between the two approaches. Formulas (syntactic) correspond to events (semantic);...By "correspond", we mean "have similar substantive content. ... But substantively, they express the same thing".
>
> "This paper examines the relation between the two approaches, and shows that they are in a sense equivalent.". [Aumann (1998)] [1]

In the syntactic approach the players' reasoning is described explicitly starting from a formal language, which is able to describe exhaustively the epistemic characteristics of agents. Instead in the semantic approach this description is obtained through a set-theoretic framework. If we consider the semantic representation just as one possible interpretation of the expressions of the formal language, the problem of the equivalence between syntactic and semantic approach is equivalent to the uniqueness of the interpretation of a formal language. Consequently, we will focus on the problem of interpretation of a language and we will borrow techniques from mathematical logic and model theory.

Once we have answered this question we turn to the following: "*whether, in what sense, and why the space* Ω *and the partitions* I_i *can be taken as given as given and commonly known by the player*" [1] (see Proposition 7). More explicitly, once we have analyze when the semantic representation of a language exists and is unique, we will investigate whether the interpretation of the language can be "common knowledge" among the players.

We start considering a set N of agents, labelled by $1, 2, ..., n$, and we assume that they want to reason about an external setting, which can be described by a nonempty countable set of

[6] Namely we restrict the set of possible propositions in our formal language.

propositions, $\Phi = \{p_1, p_2, ...\}$. Usually these propositions describe basic facts about the external world, for example "it is snowing in Milan", "the price of oil reaches 20$" or "Player j chooses action a". To express statement like "Player i knows p", where p is an arbitrary element of Φ, we need to add the modal or epistemic operators $\{K_i\}_{i=1,2,...,n}$, one for each player, to our vocabulary. For example $K_3 p$ should be read as "Player 3 knows p".

Starting from primitive propositions belonging to Φ we can build more complicated formulas by taking negation (in symbol $\neg$), conjunction (in symbol $\wedge$) and composition with respect to epistemic operators. Thus, if p_1 and p_2 are primitive propositions $\neg p_1$, $p_1 \wedge p_2$ and $K_i p_1$ are formulas of our language[7].

In this section, we will allow to form only finite sequences of symbols and we denote the set of well-formed formulas as $\mathcal{L}_n(\Phi)$ or for short $\mathcal{L}$. In this manner we have described the syntax of modal logic.

Instead of using modal logic to describe interactive reasoning of the players we consider a first-order language, which is as rich and expressive as modal logic (see [4]).

Formally a first-order language is given by specifying the following data:

1. a set of function symbols $\mathcal{F}$ and positive integers n_f for each $f \in \mathcal{F}$;
2. a set of relation symbols $\mathcal{R}$ and positive integers n_R for each $R \in \mathcal{R}$;
3. a set of constant symbols $\mathcal{C}$.

For our purpose, $\mathcal{F}$ is constituted by the countable set of unary relation symbols $\Phi^* = \{P_1, P_2, ...\}$, where each element P_k corresponds to a single primitive proposition p_k in Φ, and by the set of binary relation symbols $\{R_1, R_2, ..., R_n\}$, one for each player. Our language is purely relational and it is composed by $\langle \Phi^*, R_1, R_2, ..., R_n \rangle$.

Now, we need to define well-formed formulas, where a formula is any string of symbols built using symbols of the language, variables symbols $x_1, x_2..$, the Boolean connectives "and" (in symbol $\wedge$), "or" (in symbol $\vee$), "not" (in symbol $\neg$) and the existence and universal quantifiers (in symbols $\exists$, $\forall$). Proceeding step by step we define atomic formulas and then formulas.

[7] Moreover, instead of writing $p \vee \neg p$ we use $True$ and instead of $p \wedge \neg p$ we use $False$.

Definition 1 *ϕ is an atomic formula if ϕ is either*

1. *$x_1 = x_2$ where x_1 and x_2 are variables or;*

2. *$R_i(x_1, x_2)$ or $P_k(x_3)$, where x_1, x_2, x_3 are variables.*

Definition 2 *The set of formulas is the smallest set $\mathcal{L}^*$ containing all atomic formulas and such that*

1. *if ϕ is in $\mathcal{L}^*$ then $\neg\phi$ is in $\mathcal{L}^*$,*

2. *if ϕ and λ are in $\mathcal{L}^*$ then $\phi \wedge \lambda$ is in $\mathcal{L}^*$, and*

3. *if ϕ is in $\mathcal{L}^*$ the $\exists v_i \phi$ is in $\mathcal{L}^*$.*

A variable x can occur freely or it can be bound by a quantifier. A formula with no free variable is called a *sentence*. We denote the set of formulas of our language as $\mathcal{L}^*$.

In order to prove that this language is as expressive as modal logic, we need to construct a translation between the two languages. For each formula, $\varphi \in \mathcal{L}$, we define a corresponding first-order formula $\varphi^* \in \mathcal{L}^*$.[8]

We start from primitive propositions:

- for every primitive proposition p_k in Φ, let $p_k^* := P_k(x)$ be the corresponding first order logic unary-relation symbol with free variable x;

and then we proceed with the translation, considering negation, conjunction and epistemic expression of primitive propositions:

- for every formula in $\mathcal{L}$ of the form $\neg p_k$, let $(\neg p_k)^* := \neg P_k(x)$ be the corresponding first order formula;

- for every formula in $\mathcal{L}$ of the form $p_k \wedge q_l$, let $(p_k \wedge q_l)^* := P_k(x_1) \wedge P_l(x_2)$ be the corresponding first order formula;

[8] We adopt the convention that greek letters denote fornulas, while latin letters denote primitive expression. Starred greek letters denote first order logic formulas.

- for every formula in $\mathcal{L}$ of the form $K_i(p_k)$, let $(K_i p_k)^* := \forall x_2 (R_i(x_1, x_2) \Rightarrow p_k^*(x_2))$ be the corresponding first order formula, where x_2 is a new variable not appearing in p_k^* and $p_k^*(x_2)$ is the result of replacing all occurrences of x_1 in p_k^* by x_2.

We can complete the translation using the recursive nature of well-formed formula in a formal language. For example let $K_i(\phi)$ be a modal logic formula, where ϕ belongs to $\mathcal{L}$, its translation in $\mathcal{L}^*$ is $(K_i\phi)^*(x_1)$, that is, $\forall x_2 (R_i(x_1, x_2) \Rightarrow \phi^*(x_2))$.

Referring to modal logic, we need a semantics i.e., a model that can be used to determine whether a formula in $\mathcal{L}$ is true or false. A commonly used tool is the so-called Kripke structure. A Kripke structure M for n agents over Φ is a tuple $(S, \pi, \mathcal{K}_1, ..., \mathcal{K}_n)$, where S is a set of states, π is an interpretation that associates to each state in S a truth assignment for the primitive propositions (i.e. $\pi(s) : \Phi \to \{True, False\}$ for each $s \in S$) and $\mathcal{K}_i$ is a binary relation on S, for each $i \in \{1, 2, ..., n\}$.[9] We denote the class of all Kripke structures for n agents over Φ as $\mathcal{M}_n(\Phi)$ or briefly $\mathcal{M}$. The truthfulness of a formula in $\mathcal{L}$ depends on the state as well as the structure, hence we focus on the notion of a formula φ to be true at (M, s) (in symbols $(M, s) \models \varphi$). The relation $\models$ is defined by induction on the structure of modal language, starting from the primitive propositions.

If φ is a primitive proposition, specifically $\varphi = p$:

$$(M, s) \models \varphi \text{ if and only if } \pi(s)(p) = True.$$

If φ is a conjunction or negation of primitive propositions:

$$\begin{aligned} \text{for } \varphi &= p \wedge q,\ (M, s) \models \varphi \text{ if and only if } \pi(s)(p) = True \text{ and } \pi(s)(q) = True \\ \text{for } \varphi &= \neg p,\ (M, s) \models \varphi \text{ if and only if } \pi(s)(p) = False \end{aligned}$$

For modal expression such as $K_i p$ the definition is:

[9] Any binary relation $\mathcal{K}_k$ can be identified by a subset of $S \times S$.

for $\varphi = K_i p$, $(M,s) \models \varphi$ if and only if $(M,t) \models p$ for all t such that $(s,t) \in \mathcal{K}_i$

Similarly, given a first order language, we need a semantic or $\mathcal{L}^*$-structure to give substance to the different formulas of $\mathcal{L}^*$.

Definition 3 *An $\mathcal{L}^*$-structure, $M = \left\langle U, \{P_k^M\}_{k=1}^K, \{R_i^M\}_{i=1}^n \right\rangle$, is given by the following elements:*

1. *a non-empty set U called universe of M:*
2. *a set $R_i^M \subseteq U \times U$ for each binary relation R_i;*
3. *a set $P_k^M \subseteq U$ for each unary relation P_k.*

If a language has constant symbols $c \in \mathcal{C}$ then their interpretations are just elements c^M of U.

If φ is a formula in $\mathcal{L}^*$ with free variables $x_1, ..., x_n$ we will think of φ as expressing a property of elements of $U^n := U \times U \times ... \times U$, hence we must define what it means for $\varphi(x_1, ..., x_n)$ to hold at $(a_1, ..., a_n) \in U^n$. Once more we will use the recursive nature of the well-formed formulas.

Definition 4 *Let ϕ be a formula with free variables from $x = (x_{i_1}, ..., x_{i_m})$ and let $\bar{a} = (a_{i_1}, ..., a_{i_m}) \in U^m$. We inductively define "M satisfies $\phi(\bar{a})$" or "$\phi(\bar{a})$ is true in M" (in symbols $M \models \phi[\bar{a}]$) as follows:*

1. *if ϕ is $R_i(x_1, x_2)$ then $M \models \phi[\bar{a}]$ if $\bar{a} = (a_1, a_2) \in R_i^M$;*
2. *if ϕ is $P_k(x_1)$ then $M \models \phi[\bar{a}]$ if $\bar{a} = (a_1) \in P_k^M$;*
3. *if ϕ is $(K_i p)^* = \forall x_2 R_i(x_1, x_2) \Rightarrow P_k(x_2)$ then $M \models \phi[\bar{a}]$ if for every b, such that $(\bar{a}, b) \in R_i^M$, then $M \models P_k[b]$;*

4. *if* ϕ *is* $\neg\psi$ *then* $M \models \phi[\bar{a}]$ *if* $M \nvDash \psi[\bar{a}]$;

5. *if* ϕ *is* $(\psi \wedge \theta)$ *then* $M \models \phi[\bar{a}]$ *if* $M \models \psi[\bar{a}]$ *and* $M \models \theta[\bar{a}]$;

6. *if* ϕ *is* $\exists v_j \psi(\bar{v}, v_j)$, *then* $M \models \phi[\bar{a}]$ *if there is* $b \in M$ *such that* $M \models \psi[\bar{a}, b]$ *holds.*

7. *if* ϕ *is* $\forall v_j \psi(\bar{v}, v_j)$, *then* $M \models \phi[\bar{a}]$ *if for every* $b \in M$, $M \models \psi[\bar{a}, b]$ *holds.*

Next, we consider the mapping, proposed by Fagin et al. (1995) [4], from the set of Kripke structures, $\mathcal{M}$, to the set of $\mathcal{L}^*$-structures. This provides us with an equivalent $\mathcal{L}^*$-structure, M^*, for each Kripke structure $M \in \mathcal{M}$.

Given a Kripke structure $M = (S, \pi, \mathcal{K}_1, ..., \mathcal{K}_n)$ the corresponding $\mathcal{L}^*$-structure, M^*, has a *universe* S and for each primitive proposition $p_k \in \Phi$, we define the interpretation of the corresponding proposition $p_k^* \in \Phi^*$ to be $P_k^{M^*} = \{s \in S : \pi(s)(p) = True\}$, and the interpretation of any binary relations to be $R_i^{M^*} = \mathcal{K}_i \subseteq S \times S$.

If a primitive proposition p_k holds at some subset $S_{p_k} \subseteq S$ then the corresponding first-order expression, p_k^*, has interpretation S_{p_k}. Similarly given $\mathcal{K}_i \subseteq S \times S$, which is the semantic counterpart of the modal knowledge operator, the corresponding first order expression R_i has interpretation $\mathcal{K}_i$.

Now, we need to prove that this $\mathcal{L}^*$-structure, M^*, is equivalent to the Kripke structure in terms of truth-value. Namely, if a modal logic formula holds in a given the Kripke structure M, then its translation in the first-order language holds in the corresponding $\mathcal{L}^*$-structure, M^*.

Proposition 1 *For each formula in modal logic* φ, $(M, s) \models \varphi$ *if and only if* $M^* \models \varphi^*[s]$ *where* φ^* *is the corresponding first-order language formula.*

Proof. See Appendix A ■

Once, we have defined the language $\mathcal{L}^*$ and we have established this "valid" translation, we can consider a set of sentences that describe some properties of the epistemic operator. We define any set of sentences a $\mathcal{L}^*$-theory. Moreover, given a purely syntactic object like a theory we can investigate the class of $\mathcal{L}^*$-structures, that satisfy all the sentences of the theory.

For example, given the sentence $\forall x \varphi(x)$, where $\varphi(x)$ is a formula with just one free variable, and the $\mathcal{L}^*$-structure $M = \left\langle U, P^M, R^M \right\rangle$, if all the elements $u \in U$ satisfy the formula $\varphi(x)$, in symbols $M \models \phi[u]$, then $\left\langle U, P^M, R^M \right\rangle$ is a model for the theory consisting of the single sentence $\forall x \varphi(x)$ (in symbols $M \models \forall x \varphi(x)$).

Definition 5 *A $\mathcal{L}^*$-structure, M, is a model of T (in symbols $M \models T$) if and only if $M \models \phi$ for all sentences ϕ belonging to T.*

We will focus on a particular theory, denoted $\mathbf{T^{rst}}$, which corresponds to the Knowledge operator (or $S5$ system) in modal logic. The five axioms that characterize the epistemic operator and constitute our theory are the following $\mathcal{L}^*$-sentences:

$$\forall x_2 \left\{ \left[\forall x \left(R_i(x_1, x_2) \Rightarrow P_1(x_2) \right) \wedge \forall x_1 \left(R_i(x_1, x_2) \Rightarrow \left(P_1(x_2) \Rightarrow P_2(x_2) \right) \right) \right] \Rightarrow \forall x_1 \left(R_i(x_1, x_2) \Rightarrow P_2(x_2) \right) \right\}$$

or more compactly (Distribution Axiom)

$$\forall x_1 \left(K_i \varphi \right)^* (x_1) \wedge \left(K_i (\varphi \Rightarrow \psi) \right)^* (x_1) \Rightarrow \left(K_i \psi \right)^* (x_1)$$

$$\forall x_1 P(x_1) \Rightarrow \forall x_1 \left[\forall x_2 \left(R_i(x_1, x_2) \Rightarrow \varphi^*(x_2) \right) \right]$$

or more compactly (Knowledge Generalization Rule)

$$\text{From } \varphi^* \text{ infer } \left(K_i \varphi \right)^*$$

$$\forall x_1 \varphi^*(x_1) \Rightarrow \forall x_1 \left(K_i \varphi \right)^* (x_1)$$

$$\forall x_1 \left[\forall x_2 \left(R_i(x_1, x_2) \Rightarrow \varphi^*(x_2) \right) \right] \Rightarrow P_1(x_1)$$

or more compactly (Truth Axiom)

$$\forall x_1 \left(K_i \varphi \right)^* (x_1) \Rightarrow \varphi^*(x_1)$$

$$\forall x_1 \left[\forall x_2 \left(R_i \left(x_1, x_2 \right) \Rightarrow \varphi^* \left(x_2 \right) \right) \right] \Rightarrow \left[\forall x_3 \left(R_i \left(x_3, x \right) \Rightarrow \left(K_i \varphi \right)^* \left(x_3 \right) \right) \right]$$

$$\forall x_1 \left[\forall x_2 \left(R_i \left(x_1, x_2 \right) \Rightarrow \varphi^* \left(x_2 \right) \right) \right] \Rightarrow \left\{ \forall x_3 R_i \left(x_3, x_1 \right) \Rightarrow \left[\forall x_2 \left(R_i \left(x_3, x_2 \right) \Rightarrow \varphi^* \left(x_2 \right) \right) \right] \right\}$$

or more compactly (Positive Introspection)

$$\forall x_1 \left(K_i \varphi \right)^* \left(x_1 \right) \Rightarrow \left(K_i K_i \varphi \right)^* \left(x_1 \right)$$

$$\forall x_1 \neg \left[\forall x_2 \left(R_i \left(x_1, x_2 \right) \Rightarrow \varphi^* \left(x_2 \right) \right) \right] \Rightarrow \left[\forall x_3 \left(R_i \left(x_3, x_1 \right) \Rightarrow \neg \left(K_i \varphi \right)^* \left(x_3 \right) \right) \right]$$

$$\forall x_1 \neg \left[\forall x_2 \left(R_i \left(x_1, x_2 \right) \Rightarrow \varphi^* \left(x_2 \right) \right) \right] \Rightarrow \left\{ \forall x_3 R_i \left(x_3, x_1 \right) \Rightarrow \neg \left[\forall x_2 \left(R_i \left(x_3, x_2 \right) \Rightarrow \varphi^* \left(x_2 \right) \right) \right] \right\}$$

or more compactly (Negative Introspection)

$$\forall x_1 \left(\neg K_i \varphi \right)^* \left(x_1 \right) \Rightarrow \left(K_i \neg K_i \varphi \right)^* \left(x_1 \right)$$

$$\forall x_1 \left[P_1 \left(x_1 \right) \wedge P_1 \left(x_1 \right) \Rightarrow P_2 \left(x_1 \right) \right] \Rightarrow P_2 \left(x_1 \right)$$

or more compactly (Modus Ponens)

From φ^* and $(\varphi \Rightarrow \psi)^*$ infer ψ^*

$$\forall x_1 \left[\varphi^* \left(x_1 \right) \wedge \left(\varphi \Rightarrow \psi \right)^* \left(x_1 \right) \Rightarrow \psi^* \left(x_1 \right) \right]$$

Following the notation of [4], we denote with $\mathcal{M}^{rst}$ the class of Kripke structures that satisfy the $S5$ axioms. For our purpose, we just need to recall that, given the $S5$ axioms, K_i is a reflexive, transitive and symmetric binary relation.

We want to make sure that any Kripke structure, endowed with the knowledge operator, is a model of the theory $\mathbf{T^{rst}}$. Hence, we take a generic Kripke structure, M, and we prove that the corresponding $\mathcal{L}^*$-structure, M^*, is a model for the $\mathbf{T^{rst}}$.

Proposition 2 *Given the Kripke structure $M^{rst} \in \mathcal{M}^{rst}$ then the equivalent $\mathcal{L}^*$-structure,*

M^*_{rst}, *is a model for* $\mathbf{T^{rst}}$.

Proof. See Appendix B ■

A modal logic formula, φ, is true for any Kripke structure if and only if it is true at any given world, $s \in S$ of any given Kripke structure $M = (S, \pi, \mathcal{K}_1, ..., \mathcal{K}_n) \in \mathcal{M}^{rst}$, we denote this property as $\mathcal{M}^{rst} \models \varphi$; similarly in a first order language a $\mathcal{L}$-sentence, φ^*, is a logical consequence of a set of sentences, T, if and only if it is true in any model of T, we denote this fact as $\mathbf{T^{rst}} \models \varphi^*$.

It is quite intuitive that if a modal logic formula is true in all Kripke structures satisfying the $S5$ axioms, then the corresponding first-order formula is a logical consequence of the theory $\mathbf{T^{rst}}$. We state this fact as a proposition.

Proposition 3 $\mathcal{M}^{rst} \models \varphi$ *iff* $\forall x \varphi^* (x)$ *is a logical consequence of* $\mathbf{T^{rst}}$, *in symbols* $\mathbf{T^{rst}} \models \varphi^*$

Proof. This will be a corollary of Proposition 4. ■

3 Interpretation in game theory

We identify the *semantic approach* with the set of $\mathcal{L}^*$-structure, M^*_{rst}, corresponding to some Kripke structure $M^{rst} \in \mathcal{M}^{rst}(\Phi)$, and we denote this class as $\mathcal{M}^*_{rst}(\Phi)$ or shortly $\mathcal{M}^*_{rst}$.

We need to determine whether the *syntactic approach* can be identified in term of $\mathcal{L}^*$-structures. This is possible through the so-called Henkin construction (see [3] or [12]).

Given our first-order language $\mathcal{L}^*$ and the theory $\mathbf{T^{rst}}$ we can consider a richer language $\mathcal{L}^*_1 \supseteq \mathcal{L}^*$, constructed in such a way that for each $\mathcal{L}^*$-formula with one free variable, $\varphi^*(x)$, there is an $\mathcal{L}^*_1 - constant$, c_φ, such that $\mathbf{T^{rst}} \models (\exists x \varphi^*(x)) \Rightarrow \varphi^*(c_\varphi)$.

For example take the formula $\exists x (K_i \varphi)^* (x)$, meaning that player i could know φ^*, then we consider a constant symbol c_{φ_i} such that the sentence $(K_i\varphi)^* \left(c_{\varphi_i}\right)$ is true. The intuition is that the constant witnesses the possibility that a player i could know φ^*. It is apparent that the Henkin construction resembles the universal type space built from expressions used, for example, by Heifetz and Samet (1998) [7].

Thus, the new language is $\mathcal{L}_1^* := \mathcal{L}^* \cup \{c_\varphi : \varphi^*(x) \text{ an } \mathcal{L}^* - formula \text{ with one free variable}\}$ and for each $\mathcal{L}^*$-formula $\gamma^*(x)$ if we denote the corresponding $\mathcal{L}_1^*$-sentence $\Gamma^* := (\exists x \varphi(x) \Rightarrow \phi(c_\varphi))$, we define a $\mathcal{L}_1^*$-theory T_1 as $\mathbf{T}_1 := \mathbf{T^{rst}} \cup \{\Gamma^* : \gamma^*(v) \text{ is an } \mathcal{L}^* - formula \text{ with one free variable}\}$.

Remark 1 *Given that any $\mathcal{L}^*$-structures corresponding to some Kripke structure M^{rst} is a model of T^{rst} then any finite subset of sentences of T^{rst} has a model. Therefore any finite subset of $\mathbf{T}_1$ has a model.*

Definition 6 *An $\mathcal{L}$-theory $\mathbf{T}$ is finitely satisfiable if every finite subset of $\mathbf{T}$ admits a model or equivalently every finite subset of $\mathbf{T}$ is satisfiable.*

Now, we iteratively construct a sequence of languages, $\mathcal{L}^* \subseteq \mathcal{L}_1^* \subseteq \mathcal{L}_2^*, ...,$ and a sequence of finitely satisfiable $\mathcal{L}_i^*$-theories, $\mathbf{T^{rst}} \subseteq T_1 \subseteq T_2 \subseteq ....$ If $\varphi(x)$ is a $\mathcal{L}_i^*$-formula, then there is a constant symbol $c_\varphi \in \mathcal{L}_{i+1}^*$. c_φ such that $\mathbf{T_{i+1}} \models (\exists x \varphi(x)) \Rightarrow \varphi(c_\varphi)$. We define $\mathcal{L}_\infty^* := \bigcup_{i=1}^{\infty} \mathcal{L}_i^*$ and $\mathbf{T}_\infty := \bigcup_{i=1}^{\infty} \mathbf{T}_i$. If we take any $\mathcal{L}_\infty^*$-formula $\phi(x)$ with one free variable x then there is a constant symbol $c \in \mathcal{L}_\infty^*$ such that, by construction, $\mathbf{T} \models (\exists x \phi(x) \Rightarrow \phi(c))$. This property is called the witness property of the theory.

Definition 7 *An $\mathcal{L}^*$-theory $\mathbf{T}$ has the* witness property *if whenever $\phi(x)$ is a $\mathcal{L}^*$-formula with one free variable x, then there is a constant symbol $c \in \mathcal{L}^*$ such that $\mathbf{T} \models (\exists x \phi(x) \Rightarrow \phi(c))$.*

If a formula is not excluded by the theory then there is a constant that represents it. Given that the set of well-formed formula is countable then $\mathbf{T}_\infty$ has at most a countable number of sentences. Moreover, by construction any finite subset of $\mathbf{T}_\infty$ admits a model.

Definition 8 *An $\mathcal{L}$-theory $\mathbf{T}$ is* maximal *if for any sentence ϕ either $\phi \in \mathbf{T}$ or $\neg\phi \in \mathbf{T}$.*

We can ensure that there is a finitely satisfiable $\mathcal{L}_\infty^*$-theory, $\mathbf{T}'_\infty \supseteq \mathbf{T}_\infty$ such that for every $\mathcal{L}_\infty^*$-sentence, ϕ, either $\phi \in \mathbf{T}$ or $\neg\phi \in \mathbf{T}$ i.e., there exists a maximal finitely satisfiable $\mathcal{L}_\infty^*$-theory (see Theorem 1 below). This is a consequence of the following theorem (refer to [12]).

Theorem 1 *If* $\mathbf{T}$ *is a finitely satisfiable* $\mathcal{L}$*-theory then there is a maximal finitely satisfiable* $\mathcal{L}^*$*-theory* $\mathbf{T}' \supseteq \mathbf{T}$*.*

Once we have defined this fully encompassing language and theory, we can consider the "canonical model", denoted as $M^{canonical}$, of $\mathbf{T}'_\infty$. This model is a $\mathcal{L}^*_\infty$-structure and its universe is $U := \mathcal{C}/\sim$, where $\mathcal{C}$ is the set of constant symbols of $\mathcal{L}^*_\infty$. For any $c, d \in \mathcal{C}$, $c \sim d$ if and only if $\mathbf{T}'_\infty \models c = d$. Thus, the universe is the set of equivalence classes of $\mathcal{C}$ with respect to $\sim$. The interpretation of the constant symbol $c \in \mathcal{C}$ is $c^{M^{canonical}} = c^*$, where c^* denotes the equivalence class of c.

The interpretation of the relations symbols of $\mathcal{L}^*_\infty$ are respectively:

$$\begin{aligned} P_k^{M'_\infty} &= \left\{c_1^* : P_k\left(c_1^*\right) \in \mathbf{T}'_\infty\right\} \text{ for } k = 1, 2, ... \\ R_i^{M'_\infty} &= \left\{\left(c_1^*, c_2^*\right) : R_i\left(c_1^*, c_2^*\right) \in \mathbf{T}'_\infty\right\} \text{ for } i = 1, 2, ..., n \end{aligned}$$

$P_k^{M'_\infty}$ and $R_i^{M'_\infty}$ are well-defined. If $c^* \sim d^*$ then $P\left(c^*\right) \in \mathbf{T}'_\infty$ if and only if $P\left(d^*\right) \in \mathbf{T}'_\infty$. Note that if $c^* \sim d^*$ then $c^* = d^*$ belongs to $\mathbf{T}'_\infty$. Hence if $P\left(c^*\right) \in \mathbf{T}'_\infty$ then $P\left(d^*\right)$ is the logical consequence of $c^* = d^*$ and $P\left(c^*\right) \in \mathbf{T}'_\infty$. By maximality of $\mathbf{T}'_\infty$, it implies that $P\left(d^*\right) \in \mathbf{T}'_\infty$. A similar reasoning applies to the binary relations.

Finally, we can defined a $\mathcal{L}^*_\infty$-structure, called canonical model of $\mathbf{T}'_\infty$:

$$M^{canonical} := \left\langle U, \left\{P_k^{M'_\infty}\right\}_{k \geq 1}, \left\{R_i^{M'_\infty}\right\}_{i=1}^{n} \right\rangle.$$

Instead of proving that this is a model for $\mathbf{T}'_\infty$ and hence of $\mathbf{T}$ ($M^{canonical} \models \mathbf{T}_\infty$) we will prove that the canonical model, namely the syntactic approach, and the set of $\mathcal{L}^*$-structures, corresponding to some Kripke structures, namely the semantic approach, have the same theory.

Given a $\mathcal{L}^*$-structures M we denote with $Th\left(M\right)$ the maximal set of sentences which are satisfied by M. Using this notation we can state formally the previous statement.

Proposition 4 $\bigwedge_{M^{rst} \in \mathcal{M}^{*}_{rst}(\Phi)} Th\left(M^{rst}\right) = Th\left(M^{canonical}\right)$

Proof. One direction ($\subseteq$) is obvious once we have noted that $M^{canonical} \in \mathcal{M}^{*}_{rst}(\Phi)$. Given that $Th\left(M^{rst}\right) \supseteq Th\left(M^{canonical}\right)$ and $M^{canonical}$ is consistent with respect to M^{rst}, let Σ be a $\mathcal{L}$-sentence, if $Th\left(M^{canonical}\right) \models \Sigma$ then $Th\left(M^{rst}\right) \models \Sigma$. ■

We have proved the equivalence of the two approaches. Proposition 4 recalls Aumann's result and the result of soundness and completeness in modal logic. But, there is one aspect that makes this proof easier to be interpreted. If we think of two game theorists who try to use one of the two approaches for analyzing a strategic situation and they want to avoid any hidden assumption, then Proposition 4 tells us that the two approaches are equivalent only if we consider all the possible models of our initial theory. In fact proposition 4 points out that if we consider a specific Kripke structure we can have different theories from $Th\left(M^{canonical}\right)$.

Moreover, from this perspective, it is clear that the universe in the canonical model is just one of the possible interpretations. On this issue Aumann stated that what is problematic is the "*vocabulary of our players*".

This fact becomes crucial when we try to model a situation where players can communicate using language $\mathcal{L}^*$, then the interpretation of the language can not be taken for granted. Moreover, from Proposition 4 we can deduce that the assumption that players share the same interpretation of the language is at the basis of "Agree to disagree" results.

On one hand a model is a complete description of a situation since any conceivable expression is either true or false, but on the other hand a set of axioms may be a partial description of the same situation. Thus, a single model is too restrictive for a given theory and in order to identify a single theory we need more than one model.

We could be tempted to identify the canonical model with the so-called universal type space, in the next section we will be more precise about this and we will point out some caveat.

4 Equivalence between type spaces and $\mathcal{L}^*$-structures

In the economic literature it is an important question whether or not there exists a universal type space and whether it is well defined. The major results were found by Boge and Eisele (1979), Mertens and Zamir (1985) [13], Heifetz and Samet (1998) [7] and others. All the results are based on a hierarchical construction that finds its intuition in the linguistic description of all order of uncertainty, that players may face upon reasoning on a strategic situation. From our point of view, the length of the reasoning, made by players, is constrained by the language used by them or by the researcher. Once we have decided the language to use, the language itself will be the guidance for building the space of uncertainty and therefore the universal space.

First, we need to recall some standard concept used in game theory for modelling incomplete information settings. Given a set of states of nature Φ and a set of players $I = \{1, 2, ..., n\}$, we denote with I_0 the set of players including the external state (see [8]).

Definition 9 *A type space on Φ is a pair $\left\langle (T_i)_{i\in I_0}, (m_i)_{i\in I} \right\rangle$ or for short $\langle T, m\rangle$ where:*

1. *$T_0 = \Phi$ and T_i, for $i \in I$ is a measurable space;*
2. *for each $i \in I$, m_i is a measurable function $m_i : T_i \to \Delta(T)$ and m_0 is the identity map;*
3. *for each $i \in I$ and $t_i \in T_i$, the marginal of $m_i(t_i)$ on T_i assigns a probability equal to 1 to t_i.*

One should note that a type space is a Kripke structure, or equivalently a $\mathcal{L}^*$-structure. Indeed, given a type space $\left\langle (T_i)_{i\in I_0}, (m_i)_{i\in I} \right\rangle$, let T be the product set $\prod_{i\in I} T_i$, and let $\Pi_i : T \to T_i$ be the natural $i^{th} - projection$ function, then we can define $m_i \circ \Pi_i : T \to \Delta(T)$. For each $t \in T$ we consider the support of the probability measure $m_i \circ \Pi_i(t)$. The t's, belonging to the support of $m_i \circ \Pi_i(t)$, are those elements $t' \in T$ that player i thinks are possible given t. If we consider the graph of the correspondence $\mathbf{Supp}(m_i \circ \Pi_i)(\cdot) : T \to 2^T$, we can define the binary relation:

$$\mathcal{K}_i := \left\{ (t, t') \in T \times T : t' \in \mathbf{Supp}(m_i \circ \Pi_i)(t) \right\}$$

for each player i. The interpretation function, $\pi(t) : \Phi \rightarrow \{True, False\}$ for each $t \in T$, is still missing, but we can think of a degenerate version of it i.e., $\pi(t) : \Phi \rightarrow \{True\}$. [10]

Once we have noted this equivalence, we can apply another basic result in model theory, the Upward Löwenheim-Skolem Theorem. But, first, we need some definitions.

Definition 10 *Two $\mathcal{L}^*$-structures M^* and N^* are elementary equivalent (in symbols $N^* \equiv M^*$) if*

$$M^* \models \phi \text{ iff } N^* \models \phi$$

for all $\mathcal{L}^*$−sentences ϕ.

By definition two elementary equivalent $\mathcal{L}^*$-structures have the same theory.

Definition 11 *If M^* and N^* are $\mathcal{L}^*$-structures, with universes U and W, respectively. An $\mathcal{L}^* - embedding$ $\eta : M^* \rightarrow N^*$ is a one to one map $\eta : U \rightarrow W$ that preserves the interpretation of all the symbols of $\mathcal{L}$:*

1. $\eta\left(f^{M^*}\left(a_1, ..., a_{n_f}\right)\right) = f^{N^*}\left(\eta\left(a_1\right), ..., \eta\left(a_{n_f}\right)\right)$ *for all function symbols $f \in \mathcal{F}$ and $a_1, ..., a_{n_f} \in M^*$;*

2. $\left(a_1, ..., a_{n_R}\right) \in R^{M^*}$ *iff* $\left(\eta\left(a_1\right), ..., \eta\left(a_{n_R}\right)\right) \in R^{N^*}$ *for all relation symbols $R \in \mathcal{R}$ and $a_1, ..., a_{n_R} \in M^*$;*

3. $\eta\left(c^{M^*}\right) = c^{N^*}$ *for all constant symbols $c \in \mathcal{C}$.*

If $\mathcal{L}^* - embedding$ η is bijective then η is called an $\mathcal{L} - isomorphism$.

Any isomorphism preserves the validity of $\mathcal{L}^*$-sentences i.e., given a sentence holding in M^* then if we consider its image through η, the sentence still holds in N^*. The last definition that we need is the following:

[10] This recalls the distinction between Aumann type space, or Kripke frame, and the fully-fledged Kripke structure.

Definition 12 *If* M^* *and* N^* *are* $\mathcal{L}^*$*-structures, then an* $\mathcal{L}^* - embedding,\ j : M^* \to N^*$ *is called an elementary embedding if*

$$M^* \models \phi\left[a_1, ..., a_n\right] \text{ iff } N^* \models \phi\left[j\left(a_1\right), ..., j\left(a_n\right)\right]$$

for all $\mathcal{L}^*$*-formula and all* $a_1, ..., a_n \in M^*$*.* M^* *and* N^* *are called elementary equivalent.*

The cardinality (in symbols $|\mathcal{L}^*|$) of a first order language is equal to the number of well-formed formula in the language; similarly the cardinality of a $\mathcal{L}$-structure M^* (in symbols $|M^*|$) is the cardinality of its universe. Now we can state the Upward Löwenheim-Skolem Theorem.

Theorem 2 (Upward Löwenheim-Skolem Theorem) *Let* M^* *be an infinite* $\mathcal{L}^*$*-structure and* k *be an infinite cardinal* $k \geq |\mathcal{L}^*| + |M^*|$*. Then there is an* $\mathcal{L}^*$*-structure,* N^**, of cardinality* k *and an* $\mathcal{L}^* - embedding\ j : M^* \to N^*$ *that is elementary.*

As a direct implication, given any infinite type space $\left\langle (T_i)_{i \in I_0}, (m_i)_{i \in I} \right\rangle$ there is another type space, that satisfies the same formulas and it has cardinality of at least $|M^*| + \omega$. Where $|M^*|$ is the cardinality of the type space, or equivalently of the structure. We can state this result as a proposition.

Proposition 5 *Given any type space* $M = \left\langle (T_i)_{i \in I_0}, (m_i)_{i \in I} \right\rangle$ *there is another type space,* N*, such that any formula satisfiable in* M *is satisfiable in* N*. Moreover* N *has cardinality at least* $|M| + \omega$*.*

Proof. We know from the previous constructions that M is a $\mathcal{L}^*$-structure. Given that the cardinality of our language is ω (see Halmos [5])), i.e. $|\mathcal{L}^*| = \omega$, the thesis follows from the Upward Löwenheim-Skolem Theorem (Theorem 2). ■

This result strengthens Theorem 2.3 of Heifetz and Samet ([7]). In general the minimal embedding model of M has cardinality equal to $|M| + \omega$.

This result poses the question of whether there exists a so-called universal type space i.e., a type space that could embed any other conceivable type space. As already noted, a universal type space can be associated to a $\mathcal{L}^*$-structure. Therefore, Proposition 2 applies and the universal type space itself can be strictly embedded in another type space. From a logic point of view this might not be an issue, because theorems provable in universal type space are still valid in the "larger" model. But from a game theory perspective, this observation cast some doubts on the existence of a universal type space and on the claim that the universal type space is common knowledge

In the next section we will provide sufficient conditions for the existence of a universal type. Moreover, we will characterize some of its properties.

5 Existence of a universal type space and its properties

In this section we will study whether a so called universal type space exists. First, we need to state formally what a universal type space is. There has been two kinds of definition one constructive (see [13]) and the other implicit (see [7]). The former relies on the informal description of player higher order of uncertainty, instead the latter takes a purely set theoretic perspective. In order to be more precise about the second approach we need some preliminary definitions.

Definition 13 *Let $\langle T, m\rangle$ and $\langle T', m'\rangle$ be two type spaces on Φ, let $(\varphi_i)_{i\in I_0}$ I_0-tuple of measurable functions $\varphi_i : T_i \to T'_i$. The induced function $\varphi : T \to T'$ is called a type morphism if:*

1. *φ_0 is the identity on Φ;*
2. *for each $i \in I$, $m' \circ \varphi_i = \left(m_i \circ \varphi^{-1}\right)$.*

In other words, a type morphism embeds $\langle T, m\rangle$ in $\langle T', m'\rangle$ and preserves the epistemic characteristics of the players. If we look at type spaces in terms of $\mathcal{L}^*$-structures, then a type morphism is just a $\mathcal{L}^*$-elementary embedding between the two. As already mentioned, a type morphism does preserve for sure the property of any epistemic operator.

Definition 14 *A type space* $\langle T^*, m^* \rangle$ *on* Φ *is universal if for every type space* $\langle T, m \rangle$ *on* Φ *there is a unique type morphism from* T *to* T^*.

Heifetz and Samet (1998) [7] proved that for any measurable space Φ there exists a unique universal type space on Φ. In a companion paper Heifetz and Samet (1999) [8] gave an example of a coherent hierarchy of beliefs that could not be extended to a belief on the space of all coherent hierarchies. The parallelism, therefore, between the explicit and the implicit construction of epistemic characteristics of players breaks down. In fact the universal type space still exists but it does not coincide with the hierachical construction.

Similar results were obtained by the same authors for the so-called knowledge type space on Φ (Heifetz and Samet (1998) and (1999) [6] [9]).

Definition 15 *A knowledge type space on* Φ *is the triple* $\left\langle \Omega, \Theta, (t_i)_{i \in I} \right\rangle$*, where* Ω *is a non-empty set whose elements are called states of the world,* $\Theta : \Omega \to \Phi$ *specifies for each state of the world the state of nature that prevails there, and for each player* $i \in I$*,* t_i *is a type function from* Ω *to* $\Delta(\Omega)$ *such that the support of* $t_i(\omega)$ *is* $\Pi_i(\omega)$*.*[11]

As already anticipated, Heifetz and Samet (1998) [6] and Fagin (1994) provided an example where the complete hierarchical description of the epistemic properties of agents is unbounded. Namely, if there are at least two players and at least two states of nature then there is no universal knowledge space.

Thus, the knowledge spaces, as ordinal numbers (Halmos (1974) [5]), are unbounded, and in general there is no knowledge space in which we can embed any conceivable knowledge space.

We want to analyze these bunch of results considering as a building block, the language used to describe the state of nature and players' reasoning. In order to do that we, first, make some remarks.

We note that if the external state is uncountable then we need an uncountable set of unary relation symbols and this drives us away from standard first-order logic. Similarly if we want to

[11] Π_i is a partition of Ω, hence $\Pi_i(\cdot)$ maps Ω to a subset of the power set of Ω.

study interactive reasoning and common knowledge we need at least a countable conjunction of formulas, hence we have another reason to depart from standard first-order languages.

On the other hand if we assume that the state of nature is definable with a finite set of properties then measure theoretic problems, which are the basis of the results of Heifetz and Samet (1999) [8], are excluded.

Moreover, unbounded regression, for describing players' reasoning, is excluded by definition if we consider just first order languages. Actually if we focus on the linguistic perspective there is obviously a complete description of players' epistemic characteristics, because well-founded formulas are at most countable[12].

In order to fill this gap we need to enrich our language and we refer to the infinitary logic (see Keisler (1991) [11]).

Infinitary logic will, also, allow us to express concept like common knowledge without introducing new symbols and we will be able to determine whether there is some connections between properties of the knowledge operator and properties of common knowledge.

The first guess would be that we need a language which admits countable conjunctions of formulas, referring for example to the definition of common knowledge in modal logic. We will show that this guess is wrong and we need an even richer language.

In the previous section we have defined the equivalent formula for the epistemic expression $K_i\varphi$, that is $(K_i\varphi)^*(x_1) := \forall y\,(R_i(x_1,x_2) \Rightarrow \varphi^*(x_2))$ or more explicitly $(K_i\varphi)^*(x_1) := \forall x_2\,(\neg R_i(x_1,x_2) \vee \varphi^*(x_2))$. If we want to say that player i knows that player j knows the proposition φ, in symbols $K_iK_j\varphi$, the following $\mathcal{L}^*$-formula is needed :

$$
\begin{aligned}
(K_iK_j\varphi)^*(x_1) &= \forall y\,\{R_i(x_1,x_2) \Rightarrow (K_j\varphi)^*(x_2)\} = \\
&= \forall y\,\{\neg R_i(x_1,x_2) \vee [\forall x_3 \neg R_i(x_2,x_3) \vee \varphi^*(x_3)]\}
\end{aligned}
$$

[12] Sophistication can be expressed up to an at most countable order, but if we look at the semantic counterpart we might go "deeper".

Given this simple example, it becomes apparent that we need a language that allows for countable conjunction and countable quantification, in particular we will use the logic denoted with $\mathcal{L}_{\omega_1,\omega_1}$[13], which is built from first order language by allowing countably[14] infinite disjunctions, conjunctions and quantifiers.

As a first result, we will encounter all the difficulties related to this kind of formal languages, for example the lack of compactness, existence of a model and completeness.

Given the our language $\mathcal{L}^* = \left\langle \{P_k\}_{k\geq 1}, \{R_n\}_{n\in N} \right\rangle$, where P_k is a unary relation symbol, corresponding to one of the "external" state characteristic and R_n is a binary relation symbol, one for each player $n \in N$.

The language $\mathcal{L}^*_{\omega_1,\omega_1}$ has the same logic symbols as $\mathcal{L}^*$ but the conjunction symbol $\wedge$ can be applied to a countable or finite set of formulas, and, similarly, the quantification symbols $\forall$, $\exists$ can be applied to a countable or finite set of variables.

Definition 16 *We denote with $\mathcal{F}^*_{\omega_1,\omega_1}$ the smallest class of formulas such that:*

1. *$R_n(x,y)$ and $P_k(z)$, where x, y, z are variable symbols, belong to $\mathcal{F}^*_{\omega_1,\omega_1}$;*

2. *if ϕ belongs to $\mathcal{F}^*_{\omega_1,\omega_1}$ and the at most countable set of variable symbols[15] $\{v_\alpha\}$ then $\neg\phi$ and $\forall \{v_\alpha\} \phi$ belong to $\mathcal{F}^*_{\omega_1,\omega_1}$;*

3. *if $\Phi = \{\phi_1, \phi_2, ...\}$ is a finite or countable, non-empty, subset of $\mathcal{F}^*_{\omega_1,\omega_1}$ then $\bigwedge_{i\geq 1} \phi_i$ belongs to $\mathcal{F}^*_{\omega_1,\omega_1}$.*

Once we have defined the set of well-formed formulas $\mathcal{F}^*_{\omega_1,\omega_1}$, we enrich the rules of inference, similarly to what is done for the language $\mathcal{L}^*_{\omega_1,\omega}$[16] (see [11]):

- from φ and $\varphi \Rightarrow \theta$ infer θ (Modus Ponens in $\mathcal{L}^*_{\omega_1,\omega_1}$)[17];

[13] ω_1 is the first ordinal number bigger than ω.

[14] To be precise we should allow for uncountable disjunctions and conjunctions to capture completely the result on knowledge space, but we it will become clearer below why this is unnecessary.

[15] $|\{v_\alpha\}| \leq \omega_1$

[16] $\mathcal{L}^*_{\omega_1,\omega}$ is a formal language where it is not allowed for countable quantification

[17] Note that φ and θ belong t $\mathcal{F}^*_{\omega_1,\omega_1}$, hence in a trivial sence this is a generalization of the Modus Ponens.

- from $\varphi \Rightarrow \theta(x, ...)$ infer $\varphi \Rightarrow \forall x \theta(x, ...)$ where the variable x does not occur free in θ (Generalization);
- from $\varphi \Rightarrow \theta$ for all θ belonging to the at most countable set of formulas Θ infer $\bigwedge \Theta$ ($\mathcal{L}^*_{\omega_1,\omega_1}$−inference).

Now, we are able to define mutual knowledge $(E\varphi)^*(x) := \bigwedge_{n \in N} K\varphi^*(x)$ and common knowledge $(CK\varphi)^*(x) := \bigwedge_{k \geq 1} \left[(E\varphi)^*\right]^k(x)$ with well-formed formulas in the richer language $\mathcal{L}^*_{\omega_1,\omega_1}$. We turn to the question of whether there exists a canonical type space or canonical model of our theory, similarly to what we have done for the finitary case. Unfortunately the answer is negative. This follows from the incompleteness of our infinitary language, because in $\mathcal{L}^*_{\omega_1,\omega_1}$ there are valid formulas that are not deducible from the above axiom scheme[18]. Hence the construction of the so called canonical model stops at its very beginning because we are not able to determine whether a sentence follows from our theory $\mathbf{T^{rst}}$.

Proposition 6 *There is not a canonical model such that each state of the world represents the maximal set of consistent formulas.*

Proof. See Appendix C. ■

A possible solution is to take the set of all the formulas of $\mathcal{L}^*_{\omega_1,\omega}$, which is a subset of $\mathcal{F}^*_{\omega_1,\omega_1}$, and consider its closure with respect to the following two axioms: [19]

$$\forall x \bigwedge_{k \geq 1} (K\phi_k)^*(x) \Rightarrow \left(K \bigwedge_{k \geq 1} \phi_k\right)^*(x) \qquad \text{(Epistemic Continuity)}$$

for any $\{\phi^*_k\}_{k \geq 1}$ belonging to $\mathcal{L}^*_{\omega_1,\omega}$

and

[18] $\mathcal{L}^*_{\omega_1,\omega_1}$ is incomplete as a consequence of Scott's indefinability theorem (see [Bell 2000]) and the set of valid sentences for $\mathcal{L}^*_{\omega_1,\omega}$ is not recursively enumerable as in first order logic.

[19] The two axioms are strictly related to the Barcan property (see [10])

$$\forall x \bigvee_{k\geq 1} \neg \left(K\phi_k\right)^* (x) \Rightarrow \left(K\neg \bigwedge_{k\geq 1} \phi_k\right)^* (x) \qquad \text{(Epistemic Monotonicity)}$$

for any $\{\phi_k^*\}_{k\geq 1}$ belonging to $\mathcal{L}^*_{\omega_1,\omega}$

We denote this fragment of $\mathcal{F}^*_{\omega_1,\omega_1}$ as $\mathcal{A}^*_{\omega_1,\omega_1}$. Once we have restricted our language to a transitive closure of $\mathcal{L}^*_{\omega_1,\omega}$, we can proceed with a construction a' la Henkin and obtain a canonical space. We just restate the definition of maximal theory for this specific language:

Definition 17 *A theory $\mathcal{A}^*_{\omega_1,\omega_1}$-theory, T, is $\mathcal{A}^*_{\omega_1,\omega_1}-$ maximal if and only if for every sentence, σ, belonging to $\mathcal{A}^*_{\omega_1,\omega_1}$ either $\sigma \in T$ or $\neg\sigma \in T$.*

Lemma 1 *Suppose that a $\mathcal{A}^*_{\omega_1,\omega_1}$-theory, T, is $\mathcal{A}^*_{\omega_1,\omega_1}$-maximal and it has the witness property. If any of its finite subset admits a model then T has a model.*

Proof. See Appendix D. ■

Using the language $\mathcal{L}^*_{\omega_1,\omega_1}$, we can not start from any theory and extend it to a theory that has the witness property and then build a model for that theory, because the language $\mathcal{L}^*_{\omega_1,\omega}$ is not compact. Therefore, we can not conclude that if all the finite subsets of a theory have a model then the full theory itself has a model.

If we want a compact language we should take a fragment or a subset of $\mathcal{L}^*_{\omega_1,\omega}$, but this precludes the possibility of defining explicitly for example common knowledge. Note that the canonical model is uncountable because the set of $\mathcal{L}^*_{\omega_1,\omega}$-formulas is uncountable and $\mathcal{A}^*_{\omega_1,\omega_1}$ contains it.

We consider the $\mathcal{A}^*_{\omega_1,\omega_1}$-theory, $\mathbf{T}^{\mathbf{rst}}_{+}$, which is obtained from $\mathbf{T}^{\mathbf{rst}}$ adding axioms (Epistemic Continuity) and (Epistemic Monotonicity). $\mathbf{T}^{\mathbf{rst}}_{+}$ is finite and admits a model; we need to consider a maximal finitely satisfiable $\mathcal{A}^*_{\omega_1,\omega_1}$-theory that includes $\mathbf{T}^{\mathbf{rst}}_{+}$, in symbols $\mathbf{T}^{\mathbf{rst}\prime}_{+} \supseteq \mathbf{T}^{\mathbf{rst}}_{+}$.

Let I be the set of all finitely satisfiable $\mathcal{A}^*_{\omega_1,\omega_1}$-theories containing $\mathbf{T}^{\mathbf{rst}}_{+}$. If $C \subseteq I$ is a chain, ordered by inclusion, of sets of $\mathcal{A}^*_{\omega_1,\omega_1}$-sentences, then we can define the set of sentences

contained in the chain as $T_C := \bigcup\{\Sigma : \Sigma \in C\}$. If Δ is a finite subset of T_C then there is $\Sigma \in C$ such that $\Delta \subseteq \Sigma$, so T_C is a finitely satisfiable and $T_C \supseteq \Sigma$ for all $\Sigma \in C$.
Thus, every chain in I has an upper bound and by the Zorn Lemma we can find $\mathbf{T}_+^{\mathbf{rst}\prime}$ that is maximal with respect the partial order of inclusion.

Given $\mathbf{T}_+^{\mathbf{rst}\prime}$, which is finitely satisfiable, and ϕ is a $\mathcal{A}^*_{\omega_1,\omega_1}$-sentence then either $\mathbf{T}_+^{\mathbf{rst}\prime} \cup \{\phi\}$ or $\mathbf{T}_+^{\mathbf{rst}\prime} \cup \{\neg\phi\}$ is finitely satisfiable. Suppose $\mathbf{T}_+^{\mathbf{rst}\prime} \cup \{\phi\}$ is not finitely satisfiable. Then, there is a finite $\Delta \subseteq \mathbf{T}_+^{\mathbf{rst}\prime}$ such that $\neg\phi$ is a logical consequence of Δ.
For any finite subset of $\mathbf{T}_+^{\mathbf{rst}\prime}$, $\Sigma \subseteq \mathbf{T}_+^{\mathbf{rst}\prime}$, $\Sigma \cup \Delta \subseteq \mathbf{T}_+^{\mathbf{rst}\prime}$ hence $\Sigma \cup \Delta$ is finitely satisfiable. Moreover, if $\neg\phi$ is a logical consequence of the theory obtained by the union of Σ and Δ, $\Sigma \cup \Delta$, then $\Sigma \cup \{\neg\phi\}$ is satisfiable. Thus, $\mathbf{T}_+^{\mathbf{rst}\prime} \cup \{\neg\phi\}$is finitely satisfiable.

We can deduce that for any sentence ϕ either $\mathbf{T}_+^{\mathbf{rst}\prime} \cup \{\phi\}$ or $\mathbf{T}_+^{\mathbf{rst}\prime} \cup \{\neg\phi\}$, consequently we have found a maximal theory that contains our original theory, $\mathbf{T}_+^{\mathbf{rst}}$. By Lemma 1 we can conclude that the canonical space exists. Moreover, the canonical model is also a universal space given that Theorem 2 does not hold in $\mathcal{L}^*_{\omega_1,\omega}$. In other words, the obtained universal type space is unique because the Theorem 2 does not apply.
We can refer to Axioms Epistemic Continuity and Epistemic Monotonicity to conclude that the set of all the finite descriptions of epistemic properties of a player determine completely his epistemic type. Said that we can conclude that the canonical model is our universal type space which includes any other conceivable type space and which can not be embedded in strictly bigger type space.

Now, we investigate some properties of the universal type space.

Lemma 2 *The universal or canonical type space is compact.*

Proof. See Appendix E. ■

This result tells us that Axioms Epistemic Continuity and Epistemic Monotonicity not only guarantee the existence of a universal type space but they justify the widespread use of a compact space in the construction of it (see [13]).

At last, the following proposition makes precise in what sense the partitions could be considered common knowledge (see Aumann (1967)).

Proposition 7 *Given the Epistemic Continuity Axiom there exists a formula $\varphi^*(x)$ such that:* [20]

$$\forall x \varphi^*(x) \Rightarrow (K\varphi)^*(x)$$

Proof. Take $(CK\phi)^*(x)$ then by definition of common knowledge and the truth axiom $\forall x \left([E\varphi]^*\right)^{k+1}(x) \Rightarrow \left[K\left(E\varphi\right)^k\right]^*(x)$. Therefore $\forall x \bigwedge_{k\geq 1}\left([E\varphi]^*\right)^{k+1}(x) \Rightarrow \bigwedge_{k\geq 1} K\left([E\varphi]^*\right)^k(x)$ by the third inference rule that we have introduced in (5) and finally by the Epistemic Continuity Axiom (Epistemic Continuity) we can conclude that $\forall x \bigwedge_{k\geq 1} K\left([E\varphi]^*\right)^k(x) \Rightarrow K\left(\bigwedge_{k\geq 1}\left([E\varphi]\right)^k\right)^*(x)$.

Hence $\forall x (CK\phi)^*(x) \Rightarrow (K(CK\phi))^*(x)$. ∎

We can conclude that the canonical space is well-behaved. By construction it is sufficient for describing epistemic condition of players. Moreover, there is at lest one self-evident event.

6 Conclusions

Using techniques of model theory[21], we show that the semantic and syntactic approaches are equivalent if we want to describe the epistemic environment up to a finite order of sophistication. Once we allow for an uncountable degree of sophistication by agents or we want to define explicitly common knowledge, this parallelism breaks down. We identify sufficient condition in order to get again a complete equivalence between the two approaches and as a direct consequence we can prove the existence of the so-called universal type space. Finally we state the main properties of the resulting universal type space.

[20] We suppress the player's index for the knowledge operator because the following sentence holds for any player.

[21] Aumann ([1]) already proved this by means of modal logic.

Appendix A

Proof of Proposition 1

Proposition 8 (1) *For each formula in modal logic* φ, $(M,s) \models \varphi$ *if and only if* $M^* \models \varphi^*[s]$ *where* φ^* *is the corresponding first-order language formula.*

Proof. (Step 1) Take a primitive proposition $p \in \Phi$, $(M,s) \models p$ iff $\pi(s)(p) = True$.

Given $p^* := P(x)$ and the corresponding $\mathcal{L}^*$-structure M^*, we need to prove that $(M,s) \models p$ if and only if $M^* \models P[s]$, but $M^* \models P[s]$ holds if and only if $s \in P^M$, i.e. $\pi(s)(p) = True$.

Let φ be $K_i p$ then $(M,s) \models K_i p$ iff $(M,t) \models p$ for every t such that $(s,t) \in \mathcal{K}_i$, this is equivalent (by definition of p^* and R_i^M) to $M^* \models P[t]$ for every t such that $M^* \models R_i[s,t]$, by definition[22] $M^* \models (K_i\phi)^*[s]$.

By induction over well-formed formulas in modal logic (or equivalently over well-formed formulas in $\mathcal{L}^*$) the proof is completed.

(Step 2) $(M,s) \models \varphi$ for every state s of a Kripke structure if and only if $M^* \models \forall x \varphi^*(x)$

A formula φ is valid in M if and only if $(M,s) \models \varphi$ for every $s \in S$, i.e. $\pi(s)(\varphi) = True$ for every $s \in S$ but, by the previous step, this is equivalent to $M^* \models \varphi^*[s]$ for every $s \in S$, or equivalently $M^* \models \forall x \varphi^*(x)$.

(Step 3) $M \models \varphi$ for every $M \in \mathcal{M}$ if and only if $\forall x \varphi^*(x)$ is a valid first-order formula.

Suppose that $\forall x \varphi^*(x)$ is not a valid first-order formula, then by definition there must be a $\mathcal{L}^*$-structure, M^*, such that $\neg\forall x \varphi^*(x)$ is satisfiable, i.e. there exist $s \in S$ such that $M^* \models \neg\varphi^*[s]$; hence, by (Step 1), $(M,s) \models \neg\varphi$ (contradiction).

If $\forall x \varphi^*(x)$ is a valid first-order formula, then it holds for all $\mathcal{L}^*$-structures.
Given that the mapping is defined for every Kripke structure M, we can conclude that $M \models \varphi$ for $M \in \mathcal{M}$. ■

Appendix B

[22] In other words $M^* \models (K_i\phi)^*[s]$ iff $M^* \models \forall x (R_i([s], x) \implies \phi^*(x))$ iff for every t such that $(s,t) \in R_i^M$ then $\phi^*([t]) \in P^M$

Proof of Proposition 2

Proposition 9 (2) *Given the Kripke structure $M^{rst} \in \mathcal{M}^{rst}$ then the equivalent $\mathcal{L}^*$-structure, M^*_{rst}, is a model for $\mathbf{T^{rst}}$.*

Proof. We consider each axiom of our theory at the time:

$$\forall x \left(K_i\varphi\right)^* (x) \wedge \left(K_i \left(\varphi \Rightarrow \psi\right)\right)^* (x) \Rightarrow \left(K_i\psi\right)^* (x)$$

Let a be an element of S such that $M^*_{rst} \models (K_i\varphi)^* [a]$ and $M^*_{rst} \models (K_i (\varphi \Rightarrow \psi))^* [a]$, equivalently $(K_i\varphi)^* [a] := \forall y \left[R_i ([a], y) \Rightarrow \varphi^* (y)\right]$ and $(K_i\varphi)^* [a] := \forall y \left[R_i ([a], y) \Rightarrow (\varphi \Rightarrow \psi)^* (y)\right]$. Thus, if $b \in S$ and $R_i ([a], [b])$ then $\varphi^* [a]$ and $(\varphi \Rightarrow \psi)^* [b]$ by correctness of first-order logic we can conclude that $M^*_{rst} \models \psi^* [b]$.

$$\forall x \varphi^* (x) \Rightarrow \forall x \left(K_i\varphi\right)^* (x)$$

If for every $a \in S$ $M^*_{rst} \models \varphi^* [a]$ then for any $b \in S$ such that $(a, b) \in R_i^{M^*_{rst}}$, $M^*_{rst} \models \varphi^* [b]$.

$$\forall x \left(K_i\varphi\right)^* (x) \Rightarrow \varphi^* (x)$$

Let a be an element of S such that $M^*_{rst} \models (K_i\varphi)^* [a]$, i.e. $\forall y (R_i ([a], y) \Rightarrow \varphi^* (y))$, if $b \in S$ and $M^*_{rst} \models R_i ([a], [b])$ then $(a, b) \in R_i^{M^*_{rst}}$ or equivalently $\mathcal{K}_i (a, b)$ is true, given the assumption on the possibility relation φ is true in b ($M^*_{rst} \models \varphi^* [b]$); then given that $\mathcal{K}_i$ is reflexive $M^*_{rst} \models \varphi^* [a]$.

$$\forall x \left(K_i\varphi\right)^* (x) \Rightarrow \left(K_i K_i\varphi\right)^* (x)$$

Let a be an element of S such that $M^*_{rst} \models (K_i\varphi)^* [a]$, i.e. $\forall y (R_i ([a], y) \Rightarrow \varphi^* (y))$ and consider $b \in S$ such that $(a, b) \in R_i^{M^*_{rst}}$ if $c \in S$ such that $(b, c) \in R_i^{\mathcal{M}^*_{rst}}$ then by definition of $R_i^{M^*_{rst}}$ $\mathcal{K}_i (a, b)$ and $\mathcal{K}_i (b, c)$ are true; given that $\mathcal{K}_i$ is transitive $(a, c) \in R_i^{M^*_{rst}}$ and then by assumption $(\forall y (R_i ([a], y) \Rightarrow \varphi^* (y)))$ $\mathcal{M}^*_{rst} \models \varphi^* [c]$. Hence we have that for every $b \in S$ such that $(a,b) \in R_i^{M^*_{rst}}$ $M^*_{rst} \models \forall y \left(R_i\left([b], y\right) \Rightarrow \varphi^*(y)\right)$.

$$\forall x \left(\neg K_i \varphi\right)^* (x) \Rightarrow \left(K_i \neg K_i \varphi\right)^* (x)$$

Let a be an element of S such that $M^*_{rst} \models \neg \left(K_i \varphi\right)^* [a]$, i.e. there is $\tilde{a} \in S$ such that $M^*_{rst} \models R_i\left([a], [\tilde{a}]\right)$ and $M^*_{rst} \models \neg\varphi^* [\tilde{a}]$, now take $b \in S$ such that $M^*_{rst} \models R_i\left([a], [b]\right)$, i.e. $(a,b) \in R_i^{M^*_{rst}}$. Given that $\mathcal{K}_i$ is symmetric $M^*_{rst} \models R_i\left([\tilde{a}], [a]\right)$, as noticed before $\mathcal{K}_i$ is transitive hence $M^*_{rst} \models R_i\left([\tilde{a}], [b]\right)$, again by symmetry $M^*_{rst} \models R_i\left([b], [\tilde{a}]\right)$. Hence we have that for every $b \in S$ such that $(a,b) \in R_i^{M^*_{rst}}$ $M^*_{rst} \models \exists y \left[R_i\left([b], y\right) \wedge \neg\varphi^*(y)\right]$. ■

Appendix C

Proof of Proposition 6

First we state the Scoot's indefinability theorem.

Theorem 3 (Scott's Undefinability Theorem for $\mathcal{L}_{\omega_1,\omega_1}$) *The set of valid sentence for $\mathcal{L}^*_{\omega_1,\omega_1}$ is not definable in coding structure of hereditarily infinite sets $H(\omega_1)$ by any $\mathcal{L}_{\omega_1,\omega_1}$–formulas.*

Given that the canonical model is built by witnessing all the $\mathcal{L}_{\omega_1,\omega_1}$–formulas that are compatible with a theory, as a result of Scott's theorem there are formulas, more precisely sentences, for which we can not say whether they are compatible or not with any given theory.

Proposition 10 *There is not a canonical or universal model such that each state of the world represents the maximal set of consistent formulas.*

Proof. A sentence is a formulas without free variable, hence the statement follows from Scott's indefinability Theorem 3. ■

Appendix D

Proof of Lemma 1

Lemma 3 (1) *Suppose that a $\mathcal{A}^*_{\omega_1,\omega_1}$-theory, T, is $\mathcal{A}^*_{\omega_1,\omega_1}$-maximal and it has the witness property. If any of its finite subset admits a model then T has a model.*

Proof. Let $\mathcal{C}$ be the set of constant symbols of $\mathcal{A}^*_{\omega_1,\omega_1}$. For any $c, d \in \mathcal{C}$ $c \sim d$ iff $T \models c = d$. Note that our theory now includes Axioms Epistemic Continuity and Epistemic Monotonicity, hence given a countable set of formulas, $\{\phi^*_k\}_{k\geq 1}$, belonging to $\mathcal{A}^*_{\omega_1,\omega_1}$ the formula $\left(K \bigwedge_{k\geq 1} \phi_k\right)^* (x)$ has as a witness constant that is $\sim$ *−equivalent* to the witness of $\left(\bigwedge_{k\geq 1} \phi_k\right)^* (x)$.

As in the finitary case, it is easy to verify that $\sim$ is an equivalence relation.

The universe of our model will be $M = \mathcal{C}/\sim$, that is the equivalence classes of $\mathcal{C}$ modulus $\sim$.

With a little abuse of notation we denote the equivalence class with an arbitrary element c^M belonging to it, hence the interpretation of any constant symbol of $\mathcal{A}^*_{\omega_1,\omega_1}$ c is $c^M = c^*$. Next the binary relation, R_i, and the unary relation, P_k, symbols have interpretations defined as:

$$
\begin{aligned}
R_i^M &= \{(c_1^*, c_2^*) : R_i(c_1^*, c_2^*) \in T\} \\
P_k^M &= \{c_1^* : P_k(c_1^*) \in T\}
\end{aligned}
$$

This completes the description of the structure $M = \left\langle \mathcal{C}/\sim, \{P_k^M\}_{k\geq 1}, \{R_i^M\}_{i\in N} \right\rangle$.

We need to prove that for all $\mathcal{A}^*_{\omega_1,\omega_1}$–formulas $\varphi(v_1, ..., v_n)$ and $c_1, c_2, ..., c_n \in \mathcal{C}$ $M \models \varphi(c_1, c_2, ..., c_n)$ if and only if $\varphi(c_1, c_2, ..., c_n) \in T$.

By induction on formulas we start from atomic formula: take $R_i(v_1, v_2)$, the binary relation with free variables v_1, v_2, if $M \models R_i(v_1, v_2)$ then by the witness property there exists $R_i(c_1, c_2) \in T$. The same reasoning applies for P_k.

Suppose that the claim holds for an arbitrary $\mathcal{A}^*_{\omega_1,\omega_1}$–atomic formulas $\varphi(v_1, v_2)$ and $c_1, c_2 \in \mathcal{C}$, if $M \models \neg\varphi(c_1, c_2)$ then $M \not\models \varphi(c_1, c_2)$. By the induction hypothesis $\varphi(c_1, c_2) \notin T$ and

$\neg\varphi(c_1, c_2) \in T$, because T is maximal.

Conversely if $\neg\varphi(c_1, c_2) \in T$ then $\varphi(c_1, c_2) \notin T$, because T is finitely satisfiable. Thus, by inductive assumption $M \nvDash \varphi(c_1, c_2)$ and $\mathcal{M} \models \neg\varphi(c_1, c_2)$. The same way of reasoning applies for $\wedge$ and countable conjunction of formulas in $\mathcal{A}^*_{\omega_1,\omega_1}$, thanks to the completeness of $\mathcal{L}^*_{\omega_1,\omega}$.

Hence we have a canonical model for our theory. ■

Appendix E

Proof of Lemma 2

Recall that $\mathbf{T}^{\mathbf{rst}\prime}_{+}$ is the maximal, finitely satisfiable theory containing $\mathbf{T}^{\mathbf{rst}}_{+}$. We need some definition before we can prove Lemma 2.

Definition 18 *Let p be the set of $\mathcal{A}^*_{\omega_1,\omega_1}$–formulas in free variables $x_1, ..., x_n$, p is an $n-type$ if $p \cup \mathbf{T}^{\mathbf{rst}\prime}_{+}$ is satisfiable.*
p is a complete $n-type$ if for all $\mathcal{A}^*_{\omega_1,\omega_1}$-formula, φ, with free variables $x_1, ..., x_n$ either $\varphi \in p$ or $\varphi \notin p$.
Denote with S^T_n the set of all complete $n-types$ of theory $\mathbf{T}^{\mathbf{rst}\prime}_{+}$.

Take the set of *complete $n-types$* of $\mathcal{A}^*_{\omega_1,\omega_1}$ of theory $\mathbf{T}^{\mathbf{rst}\prime}_{+}$ and for $\varphi \in \mathcal{A}^*_{\omega_1,\omega_1}$, we can define:

$$[\varphi] = \{p \in S^T_n : \varphi \in p\}$$

If p is complete type and $\phi \vee \gamma \in p$, then $\phi \in p$ and $\gamma \in p$. Thus we have $[\phi \vee \gamma] = [\phi] \cup [\gamma]$. Similarly $[\phi \wedge \gamma] = [\phi] \cap [\gamma]$.

Now we can define the so-called Stone topology on S^T_n.

Definition 19 *The Stone topology on S^T_n is the topology generated by taking the sets $[\phi]$ as basic open sets.*

By definition, if p is a complete types then exactly one of ϕ and $\neg\phi$ is in p. Thus $[\phi] = S_n^T \setminus [\neg\phi]$ is also closed. Then the Stones topology is composed by sets that are both open and closed, or briefly *clopen*.

Lemma 4 *S_n^T is compact.*

Proof. We need to prove that every cover of S_n^T, by open sets, has a finite subcover.

Suppose not, then let $C := \{[\varphi_i(v)] : i \in I\}$ be a cover of S_n^T by basic open sets without any finite subcover. Let $\Sigma = \{\neg\varphi_i(v) : i \in I\}$.

We claim that $\Sigma \cup \mathbf{T}_+^{\mathbf{rst}\prime}$ is finitely satisfiable

Given the assumption that there is no finite subcover of C, if I_0 is a finite subset of I then, there is a type p such that $p \notin \bigcup_{i\in I_0} [\varphi_i(v)]$. Let N be a model of $\mathbf{T}_+^{\mathbf{rst}\prime}$, such that there is an element u of its universe satisfying all the formulas of p, i.e. there exists u such that $N \models \phi[u]$ for any formula ϕ belonging to p.

Then $N \models \mathbf{T}_+^{\mathbf{rst}\prime} \cup \bigwedge_{i\in I_0} \neg\varphi_i(u)$, so Σ is finitely satisfiable. Hence by lemma 1 Σ is satisfiable.

Hence take a model of $\mathbf{T}_+^{\mathbf{rst}\prime} \cup \mathbf{\Sigma}$ such that there is an element u' of its universe, such that $N' \models \phi[u']$ for any formula ϕ belonging to Σ.

Then the complete type $\{\varphi(v) \in \mathcal{A}^*_{\omega_1,\omega_1} : N \models \varphi(u')\} \in S_n^T \setminus \bigcup_{i\in I} [\varphi_i(v)]$, a contradiction, given the assumption that C is a cover of S_n^T. ■

Lemma 5 *The universal type space is compact.*

Proof. Take the set of *complete* $n-$*types* of $\mathcal{A}^*_{\omega_1,\omega_1}$ and for $\varphi \in \mathcal{A}^*_{\omega_1,\omega_1}$ let

$$[\varphi] = \{p \in S_n^T : \varphi \in p\}$$

As already said, the Stone topology on S_n^T is the topology generated by taking the sets $[\varphi]$ as open sets. Following Lemma 4 we can conclude S_n^T is compact moreover by the Tychonoff theorem (see Dudley 2003) $\prod_{n\geq 1} S_n^T$ is compact relatively to the product topology. ■

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