# Networks in a Bilateral Oligopoly

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## 1 Introduction

The aim of this work is to provide a preliminary approach to the analysis of the process of endogenous price formation in thin markets. In such markets, also denoted as bilateral oligopolies, both sides, being typically concentrated, have some market power, so that both buyers and sellers are able to a¤ect the prices at which they trade. Furthermore, due to the absence of serious transaction costs, traders in such markets are usually able to a¤ect at some extent the choice of their trading partners.

Examples of bilateral oligopolies may be found in basic commodities markets - such as the ones for the co¤ee, tobacco or minerals - in all the energy markets and in most the intermediate goods markets - such as the majority of the manufacturing industries, the aerospace or defence industries, the hi-tech. Furthermore, examples of thin markets emerge every time the stocks and derivatives markets are characterized by a limited number of traders.

As a few pioneering studies have recently pointed out (Bjornerstedt and Stennek (2001, 2004), Inderst and Wey (2003)), the process of price formation in bilateral oligopolies is rather peculiar. Indeed, it is very unlikely that the traders on any side of the market may behave as price-takers. Rather, it seems reasonable to think to the formation of the price as the outcome of a complex of negotiations among traders. The mentioned studies have argued that bilateral oligopolies may be reduced to a simple collection of many bilateral monopolies: the prices, thus, may emerge as the outcome of many simultaneous Nash-bargaining co-operative solutions, or of many simultaneous bilateral negotiations each involving an exogenously matched pair of one seller and one buyer.

In this paper, at the contrary, we focus on non-cooperative inter-dependent bargaining solutions where all the sellers and the buyers in the thin market can simultaneously negotiate while not being constrained by a …xed partner.

In the literature on non-cooperative bargaining in decentralized markets it is traditionally assumed that buyers and sellers are pairwise matched through some random procedure, and that the order in which agents can make or respond to price o¤ers is exogenously given.

However, as Chatterjee and Dutta (1998) observe, while these assumptions are acceptable when modelling large anonymous markets, they are less appropriate in thin markets where the search costs are usually low, and, particularly when agents are heterogeneous, traders may have interest in choosing their partner.

Chatterjee and Dutta have in fact attempted to provide a …rst insight into the e¤ect of competition for bargaining partners on the price - or, the prices - that prevail in thin markets, as well as how the matches themselves are simultaneously determined. However, they have focused on a bargaining model with alternating o¤ers between sellers and buyers. The latter not only turns out to drive rather substantially their results, but also, more importantly, may be hardly seen as a reliable description of real life negotiations out of the simple and special case of two-persons bargaining. Furthermore, the overall analysis they conduce does not seem to us particularly convincing and rigorous in the way it captures the strategic interactions among traders, the characterization of the equilibria and the conditions under which they emerge.

At the contrary, we focus on a model of negotiations with public o¤ers and a random order of proposers, which has been usually adopted for the analysis of bargaining in large decentralized markets (see for instance Rubinstein and Wolinsky (1990), Gale (1986), and, specially, De Fraja and Sakovics (2001)) for being easily comparable with the outcome of a Walrasian competitive market.

The …rst aim of our analysis is, then, to explore the strategic non-cooperative micro-foundations of price formation in markets with a limited number of traders, along the way already investigated for the case of large decentralized markets.

The second objective of the present work is the attempt to fully endogenize the matching mechanism between traders on di¤erent side of the market, by explicitly modelling the underlying individual strategical choice of the set of the eligible trading partners.

The existence of …sical infrastructural networks, altogether with their shape, in fact, play a crucial role in the distribution of bargaining power and in the feasibility of the implementation of trades among energetic companies both in national and in international gas, oil and electricity thin markets. Furthermore, most the intermediate markets are endowed with an immaterial web of communication, reputation and trust links which is very likely to a¤ect business relationships and negotiations.

Therefore, we aim in particular at drawing a preliminary picture of the interrelations among bargaining in thin markets with heterogeneous traders and strategic formation of buyers-sellers networks.

The issue of endogenous formation of trading links has been already tackled by Kranton and Minehart (2001) and Calvò-Armengol (2003). On the other hand, sound description of the negotiations' outcome given a …xed network structure has been provided by Calvò-Armengol (2001) and Corominas-Bosch (2004). From this perspective, then, our work may be seen as lying at the crossroads between these two approaches, as concerns the case of decentralized thin markets.

We attempt to provide a model of endogenous price formation in the simplest case of bilateral oligopoly where trading is restricted by endogenously formed bipartite networks. In particular, we consider completely decentralized negotiations with random order of proposers in a bilateral duopoly with heterogeneous buyers: trade of an homogeneous good between a seller and a buyer is possible only whether a link is present between them. Links structure is endogenously formed by previous non-cooperative decision by agents.

## 2 The Model

We focus on the simplest case of a bilateral duopoly, where two identical sellers, $S_1$ and $S_2$ each owns one single unit of an homegeneous indivisible good. Both sellers have the same reservation value of zero for the good. We will refer to sellers as females.

In the thin market there are two heterogeneous buyers, $B_1$ and $B_2$, both of whom demand one unit each of the commodity. The buyers' valuations are $v_1 = 1$ and $v_2 = \lambda$, respectively, with $1 > \lambda > 0$. In the following, we will sometimes refer to buyers as males, and to $B_1$ and $B_2$ also as the strong and the weak buyer, respectively.

Importantly, we assume that all the valuations are common knowledge.

The prices at which the goods are exchanged if trade takes place, are exclusively determined by endogenous bargaining among the players. In particular, we assume that all the traders in the thin market negotiate according to a public o¤ers bargaining procedure with random order of proposers.

The key feature of the model, however, is that trade may only take place between a buyer and a seller who are directly linked each other. That is when an agent $i$ on one side of the market has to respond to a price o¤er from traders belonging to the opposite side, he - or she -

may only accept or reject a proposal from $j$ such that $g_{ij} = 1$, where, as usual, $g_{ij}$ denotes the existence of a connection among agents $i$ and $j$. Analogous restrictions hold for proposal of price offers by agent $i$, which are intended to be directed exclusively to counterparts $j$ such that $g_{ij} = 1$. It is then helpful to denote with $L(i)$ the set of traders on the opposite side of the market linked with agent $i$.

We propose a two-stages game. In the first stage, the network formation stage, all the players simultaneously and independently decide which trader on the opposite side they are willing to make connection with. A buyer-seller link is formed whenever both the traders involved have chosen to form it. As the creation of a connection with a potential partner is representing in our model an approximative measure of firms' effort to invest in searching for trading opportunities, link formation is costly, at a unit rate $c < \lambda/2$.

It is immediate to see that in our thin market, only four network architectures may emerge as the outcome of the non-cooperative link formation game at the first stage. In fact all the possible networks are the empty one (Figure 1a), the structures where only one buyer-seller pair is connnected (1b), the supply-short-side networks where only one seller is linked to both buyers (1c), the demand-short-side networks where one (either weak or strong) buyer is connected to both the sellers (1d), the exclusive trade networks where each agent on any side of the market is linked only with a single partner (1e), the two asymmetrically connected structures where either the high-valuation or the low-valuation buyer is linked with both sellers, while the other buyer is connected only with one exclusive partner (1f), and, finally, the complete connected bipartite graph (1g).

At a first glance, only the two latter non-empty network structures (1f-g) may embed non trivial bargaining issues, as the for others this can be easily reduced either to a collection of several independent bilateral negotiations à la Rubinstein, or to some combination of bargaining and bidding where the short side of the market should manage to extract (almost) all the surplus from the trade.

To attempt to shed some light on the price formation in such cases we, then, need to specificy some model for the bargaining process. In fact, once the link formation game has been solved, traders enter the second stage.

In the negotiation stage, at every round $t \in \{1, 2, \dots\}$, one side of the thin market is randomly selected to propose offers: both the supply and the demand sides may be selected to be the proposers with equal probability $\frac{1}{2}$, independently by the past histories and random draws. In real thin stock markets, in fact, at any period of negotiations, there is some chance to formulate the price offers to sell or to buy at the asset.

Any round of the negotiation stage is composed by two phases. First takes place the price-offers phase: the agents on the side of the market which has been selected - for instance the buyers $i$ with $i = B_1, B_2$ - each simultaneously announce a price $p_i \in [0, 1]$ at which they are willing to buy one unit of the good, from any linked seller $j = S_1, S_2 \in L(i)$, i.e. such that $g_{ij} = 1$.

Therefter, the price-response phase occurs. Each seller $j$ responds, simultaneously and independently, to any price offer from $i \in L(j)$. A response is either acceptance of one offer $p_i$, $i \in L(j)$ or rejection of both of them.

If both sellers $j = S_1, S_2 \in L(i)$ accept $i$'s offer of $p_i$, then one of them is matched for trade with equal probability with $i = B_1, B_2$. At the contrary, both pairs are matched if the two sellers accept offers from different buyers from their set of linked partners.

Matched pairs leave the market with the good being exchanged at the agreed price offer. All the links connecting them with any of the traders are immediately removed by the bipartite graph. If, as effect of such a removal, some agents get isolated, they automatically get zero payoff and leave the market.

If, at the contrary, some not isolated pairs remain unmatched at the end of period $t$, then in period $t + 1$ the remaining traders will enter the next round of the negotiation stage with a new random selection of the market's side entitled to make proposals. No further network formation stage will ever be accessed, reflecting the fact that communication links are fixed over the time for negotiations. Such a procedure is repeated so long as some not isolated pairs remain in the market.

It is worthwhile to underline a consequence of our bargaining procedure. Once just a single buyer and a single seller have already been matched to trade and have left the market, if the two remaining traders are linked each other, the subsequent negotiation stage reduces to a standard bilateral bargaining with random order of proposers. Therefore, from the following bilateral trade, the remaining players expect a surplus of $\frac{1}{2}$ each in case the strong buyer is still in the market, or, alternatively, a surplus of $\frac{\lambda}{2}$ whenever the weak buyer is.

We assume impatience, so that all agents have a common discount rate $\delta \in [0, 1)$. Thus, if one unit of the good is exchanged in period $t$ between the buyer $i$ and the seller $j$ at the price $p$, then the payoff of the buyer will be $\delta^{t-1}(v_i - p)$ and the payoff of the seller $\delta^{t-1}p$.

By the usual backward induction argument, the overall game may be solved first for the negotiation game among traders given a fixed network structure, and then for the non-cooperative network formation game.

As already mentioned, the latter is modelled as a one-shot game where all the players simultaneously and independently decide which trader on the opposite side they want to be linked with. A buyer-seller link is formed whenever both the traders involved have chosen to form it. Hence, the resulting networks formed in the first stage correspond to the pairwise stable Nash equilibria of the game (Calvò-Armengol, 2004).

The described negotiation game is a infinite horizon dynamic game of complete and imperfect information: in fact, players' payoff functions are common knowledge and, although at each move in the game the players knows the full history of the play thus far, both the price-offers and the price-response phases in the negotiation stage are simultaneous moves games. Therefore in the following analysis we will need to solve the game for its subgame-perfect Nash equilibria. More precisely, we will look for those players' strategies, describing a complete plan of proposals in the price-offers phases and of decisions of either acceptance or rejection in the response phases, which generate a Nash equilibrium in the immediately subsequent price-response phases and which constitute a Nash equilibrium in every subgame.

In particular, given the overall complexity of the present game, we will only focus on the subgame-perfect Nash equilibria in pure and stationary strategies (PSSPE). That is, we will only consider equilibria where traders adopt pure strategies at every move, and whose strategies exclusively depend on the number of traders still active in the market and on which phase of the negotiation stage the players are. Therefore, players' strategies are not allowed to be mixed, behavioral or history dependent: any trader always proposes the same price at every equivalent node where he or she has to make an offer, and he or she always behaves in the same way whenever facing identical proposals in the price-response phase.

## 3 Bargaining in a Fixed Network

Here we solve for the bargaining sequential game between the traders in the thin markets, given the existence of a fixed bipartite network structure. The issue of which network emerges in equilibrium will be faced in the next section.

Note that the bargaining process of price formation is trivial for the cases of the empty network, where each agent takes a zero payoff, and of the one-pair and exclusive trade networks, in which the strong buyer and his matched seller get in expected terms a payoff of $\frac{1}{2}$, while the weak buyer and his matched seller each earn an expected surplus of $\frac{\lambda}{2}$.

Furthermore, whenever in the formed network one single trader remain isolated, as in the short-side networks, the negotiation stage can be reduced to a standard model of bargaining with random order of proposals either between one seller and two heterogeneous buyers, or between one (weak or strong) buyer and two identical sellers. As it is shown in a related work (Galizzi (2003)) and as it will be reported in the next Sections, our model of endogenous price formation in such cases is equivalent to combining elements of bargaining and bidding in an auction-type of setting. In particular, in the latter case, closely resembling a competition à la Bertrand, the buyer, on the short side of the market, in equilibrium appropriates all the surplus from the trade, while in the former case the overall equilibrium outcome of the negotiations depends crucially on the di¤erence between the buyers' reservation prices: whenever $\lambda$ is high enough, the good is always sold to the strong buyer for a trading price in between the two reservation values, while if $\lambda$ is low, and the weak buyer is then inin‡uent in the thin market, the equilibrium outcome is absolutely equivalent to the one of a bilateral negotiation between the seller and the strong buyer.

Thus, in the following we will start describing the negotiations in the case all the agents are connected by a complete bipartite graph, and we will then address our analysis to the bargaining process given an asymmetrically connected network, where either the strong or the weak buyer alone is linked to both sellers.

## 3.1 The Complete Bipartite Graph

In the complete bipartite network, each buyer is connected with both the sellers. The existence of links between all the possible buyer-seller pairs should enable the exploitation of any potential trade in the thin market.

The case of a bilateral duopoly connected by a complete bipartite graph corresponds to the market structure studied by the public o¤ers model in Chatterjee and Dutta (1998). In fact, in such a case, our model of negotiations di¤ers from the latter only in that the bargaining procedure follows a random, rather than an alternating, order of proposers. Furthermore, our model of negotiations in such a case may be considered as a generalization of the corresponding …ndings by Corominas-Bosch (2004) to the case of heterogeneous buyers.

Our model of negotiations among traders in such a complete bipartite graph implies that after a single buyer and a single seller have been matched to trade, have left the market and all their links have been removed, the two remaining traders have always the chance to stay in the market to carry on further negotiations. In fact they automatically access a standard bilateral bargaining with random order of proposers, whose PSSPE equilibrium payo¤s are the one described by the traditional literature: if the strong buyer is still in the market, each of the remaining traders expects from the following bargaining rounds a surplus of $\frac{1}{2}$, alternatively, whenever the weak buyer is the one left, they expect a surplus of $\frac{\lambda}{2}$.

To select a candidate for the equilibrium in the present case may not be straightforward. Economic intuition may only suggest the two sellers would compete each other in the attempt to sell to the highest-valuation buyer. Co-existence of two di¤erent prices, however, can not be a priori ruled out, as well as delays in the trade.

Indeed, it turns out that the set of the potential candidates to the PSSPE equilibrium may be e¤ectively restricted by the help of a process of sequential elimination of all the possible subgames where the strategies adopted by the traders either in the price-o¤ers or in the response phase are evidently not PSSPE equilibrium, being not immune by pro…table deviations.

First, note that there can not be any PSSPE in the negotiation stage where the bargaining process keeps on going on forever. Indeed, as the discounted payo¤s of all the traders would be zero in such a case, there is certainly a pro…table deviation at least by the strong buyer. In fact, whenever he is selected to make an o¤er, $B_1$ can always propose a price equal to $\delta$, which, being the highest price both sellers may ever gain in the following rounds, will be immediately accepted in the subsequent response phase. In turn, being $\delta < 1$ ensures the strong buyer a strictly positive payo¤, and then a pro…table deviation from the perpetual disagreement situation.

Finally, notice that, by a standard argument by theory of in…nite horizon dynamic games of complete information (see for instance Osbourne and Rubinstein (1990), Fudenberg and Tirole (1996)), a stationary dynamic game may be fully characterized by describing any of its strategically equivalent subgames.

In particular, de…ne $S$-games the subgames of the original game of negotiations among the four traders in a given network, starting whenever the sellers are randomly selected to make o¤ers. Analogously de…ne $B$-games the subgames of the original game that start when the buyers are randomly selected to make o¤ers. Hence, being all the $S$-games and all the $B$-games strategically equivalent by the stationarity hypothesis, the analysis of the PSSPE equilibria in our original dynamic game perfectly corresponds to the investigation of the PSSPE equilibria in both the $S$-games and $B$-games.

| | | **B₂** $p_1$ | **B₂** $p_2$ | **B₂** ø |
|---|---|---|---|---|
| **B₁** | $p_1$ | $\frac{1}{2}(1-p_1)+\frac{d}{4}, \frac{1}{2}(l-p_1)+\frac{dl}{4}$ | $1-p_1,\ l-p_2$ | $1-p_1,\ \frac{dl}{2}$ |
| **B₁** | $p_2$ | $1-p_2,\ l-p_1$ | $\frac{1}{2}(1-p_2)+\frac{d}{4}, \frac{1}{2}(l-p_2)+\frac{dl}{4}$ | $1-p_2,\ \frac{dl}{2}$ |
| **B₁** | ø | $\frac{d}{2},\ l-p_1$ | $\frac{d}{2},\ l-p_2$ | $dW(B_1), dW(B_2)$ |

Figure 1:

## 3.2 $S$-games

We start exploring all the subgames of the original game starting with the selection of the sellers as proposers.

By the hypothesis of stationarity, when looking for candidate PSSPE in the $S$-games we just need to consider any generic round of the negotiation stage in which sellers have been selected to make offers. Within that generic round, we then procede by backward induction, analysing first the acceptance game played by the buyers in the response phase, and then, the proposal game played by the sellers in the price-offers phase.

Therefore, denote $p_1$ and $p_2$ the prices offered simultaneously and independently by sellers $S_1$ and $S_2$, respectively, in the price-offers phase of a generic $S$-game.

We are then able to fully characterize the acceptance or rejection game played by the two buyers in the price-response phase. In fact, the strategies available to each of the responding buyers are just three: either "Accept $p_1$", or "Accept $p_2$", or, finally "Reject both prices". Hence, the normal-form representation in Figure 1 describes the response game:

By comparing the resulting payoffs and after some manipulations, we can work out a full description of the best response strategies by any buyer in the response phase.

In fact, consider first the strong buyer $B_1$. If the weak buyer plays strategy "Accept $p_1$", then the best response by $B_1$ is to also "Accept $p_1$" if

$$\begin{cases} p_1 \le 1 - \frac{\delta}{2} \\ p_1 \le 2p_2 + \frac{\delta}{2} - 1 \end{cases},$$

or, alternatively, to choose to "Accept $p_2$" if

$$\begin{cases} p_2 \leq 1 - \frac{\delta}{2} \\ p_1 \geq 2p_2 + \frac{\delta}{2} - 1 \end{cases},$$

or, finally, to "Reject both prices" if

$$\begin{cases} p_1 \geq 1 - \frac{\delta}{2} \\ p_2 \geq 1 - \frac{\delta}{2} \end{cases}.$$

On the other hand, in case the weak buyer $B_2$ plays strategy "Accept $p_2$", then the best response by $B_1$ is to "Accept $p_1$" if

$$\begin{cases} p_1 \leq 1 - \frac{\delta}{2} \\ p_2 \geq 2p_1 + \frac{\delta}{2} - 1 \end{cases},$$

or, to also "Accept $p_2$" if

$$\begin{cases} p_2 \leq 1 - \frac{\delta}{2} \\ p_2 \leq 2p_1 + \frac{\delta}{2} - 1 \end{cases},$$

or, again, to decide to reject both price offers if

$$\begin{cases} p_1 \geq 1 - \frac{\delta}{2} \\ p_2 \geq 1 - \frac{\delta}{2} \end{cases}.$$

Finally, given that the weak buyer $B_2$ plays strategy "Reject both prices", the best response by the strong buyer $B_1$ is to opt for "Accept $p_1$" if

$$\begin{cases} p_2 \geq p_1 \\ p_1 \leq 1 - \delta W(B_1) \end{cases},$$

where, again, $\delta W(B_1)$ is the discounted value of the expected payoff by entering a new round of the negotiation stage with all the four traders still active. Alternatively, the best response by the strong buyer $B_1$ is to opt for "Accept $p_2$" if

$$\begin{cases} p_1 \geq p_2 \\ p_2 \leq 1 - \delta W(B_1) \end{cases},$$

or, finally, to also decide to "Reject both prices" if

$$\begin{cases} p_1 \geq 1 - \delta W(B_1) \\ p_2 \geq 1 - \delta W(B_1) \end{cases}.$$

Analogously, we can fully characterize the set of best responses by the weak buyer in the acceptance/rejection game of the response phase.

In fact, if the strong buyer $B_1$ plays strategy "Accept $p_1$", then the best response by $B_2$ is to also "Accept $p_1$" if

$$\begin{cases} p_1 \leq \lambda\left(1 - \frac{\delta}{2}\right) \\ p_1 \leq 2p_2 + \frac{\delta\lambda}{2} - \lambda \end{cases},$$

or, alternatively, to choose to "Accept $p_2$" if

$$\begin{cases} p_2 \leq \lambda\left(1 - \frac{\delta}{2}\right) \\ p_1 \geq 2p_2 + \frac{\delta\lambda}{2} - \lambda \end{cases},$$

or, finally, to "Reject both prices" if

$$\begin{cases} p_1 \geq \lambda\left(1 - \frac{\delta}{2}\right) \\ p_2 \geq \lambda\left(1 - \frac{\delta}{2}\right) \end{cases}.$$

On the other hand, in case the strong buyer $B_1$ plays strategy "Accept $p_2$", then the best response by $B_2$ is to "Accept $p_1$" if

$$\begin{cases} p_1 \leq \lambda\left(1 - \frac{\delta}{2}\right) \\ p_2 \geq 2p_1 + \frac{\delta\lambda}{2} - \lambda \end{cases},$$

or, to also "Accept $p_2$" if

$$\begin{cases} p_2 \leq \lambda\left(1 - \frac{\delta}{2}\right) \\ p_2 \leq 2p_1 + \frac{\delta\lambda}{2} - \lambda \end{cases},$$

or, again, to decide to reject both price offers if

$$\begin{cases} p_1 \geq \lambda\left(1 - \frac{\delta}{2}\right) \\ p_2 \geq \lambda\left(1 - \frac{\delta}{2}\right) \end{cases}.$$

Finally, given that the strong buyer $B_1$ plays strategy "Reject both prices", the best response by the weak buyer $B_2$ is to opt for "Accept $p_1$" if

$$\begin{cases} p_2 \geq p_1 \\ p_1 \leq \lambda - \delta W(B_2) \end{cases},$$

where, as above, $\delta W(B_2)$ is the discounted value of the expected payoff by entering a new round of the negotiation stage when all the four traders are still active. Alternatively, the best response by the weak buyer $B_2$ is to opt for "Accept $p_2$" if

$$\begin{cases} p_1 \geq p_2 \\ p_2 \leq \lambda - \delta W(B_2) \end{cases},$$

or, finally, to also decide to "Reject both prices" if

$$\begin{cases} p_1 \geq \lambda - \delta W(B_2) \\ p_2 \geq \lambda - \delta W(B_2) \end{cases}.$$

Hence, to characterize the possible Nash equilibria of the acceptance game in the response phase, it would be sufficient to derive under which conditions a given pair of strategies by the buyers may possibly represent a mutual best response. However, even the preliminary check to whether some system of the above restrictions can describe a non-empty set in the $(p_1, p_2)$ space it turns out to be cumbersome and not significantly conclusive.

Therefore, we should better consider the overall negotiation process occurring in a generic round of a $S$-game. In fact, some of the potential Nash equilibria in the response phase turn out to be impossible to substain simply because there are no equilibria in the price-offers phase that can possibly generate such strategies in any of their subgames.

We believe an helpful starting point is to characterize all the possible PSSPE equilibria emerging in a $S$-game. In fact, exactly 9 potential equilibrium allocations of the goods may emerge in a subgame starting in a bargaining period in which the sellers make proposals. In fact, if one equilibrium does exist it must necessarily be one from the following allocations:

$$\begin{array}{cc} S_1 & S_2 \\ p_1 & p_2 \\ \emptyset & \emptyset \end{array}; \quad \begin{array}{cc} S_1 & S_2 \\ p_1 & p_2 \\ B_1 & \emptyset \end{array}; \quad \begin{array}{cc} S_1 & S_2 \\ p_1 & p_2 \\ \emptyset & B_1 \end{array}; \quad \begin{array}{cc} S_1 & S_2 \\ p_1 & p_2 \\ B_2 & \emptyset \end{array}; \quad \begin{array}{cc} S_1 & S_2 \\ p_1 & p_2 \\ \emptyset & B_2 \end{array}$$

$$\begin{array}{cc} S_1 & S_2 \\ p_1 & p_2 \\ B_1, B_2 & \emptyset \end{array}; \quad \begin{array}{cc} S_1 & S_2 \\ p_1 & p_2 \\ \emptyset & B_1, B_2 \end{array}; \quad \begin{array}{cc} S_1 & S_2 \\ p_1 & p_2 \\ B_1 & B_2 \end{array}; \quad \begin{array}{cc} S_1 & S_2 \\ p_1 & p_2 \\ B_2 & B_1 \end{array},$$

where the last row indicates the set of the buyers in $L(S_i)$ who accept the price $p_i$ proposed by the seller $S_i$, $i = 1, 2$.

In what follows, therefore, we classify the 9 possible allocations in 4 Classes, we describe the payoffs attainable in each Class and we show that some of them can never represent a PSSPE of the $S$-games. We thus gradually restrict the set of the potential equilibria to fewer classes of cases. Finally, we show that in the $S$-games there exists a PSSPE equilibrium where both sellers propose the same price.

The First Class of PSSPE emerges when, in the response phase, both buyers reject both the proposals $p_1$ and $p_2$ offered by the two sellers in the previous price-offers phase. We may represent the candidate equilibrium by the figure

$$\begin{array}{cc} S_1 & S_2 \\ p_1 & p_2 \\ \downarrow & \downarrow \end{array}.$$

In such a case all the players do not trade and enter the next round, with a new selection of the proposers. Their relative surplus are given by the discounted value of the expected payoffs by entering a new stage of negotiation.

Here we claim that this case can never constitute a PSSPE equilibrium.

**Claim 1** For $\delta > \overline{\delta}$, there is no PSSPE equilibrium in the negotiation stage where both buyers reject both price offers.

*Proof.* In the Appendix. ■

The Second Class of PSSPE emerges when only one from the two buyers accepts one price, while the other rejects both. This situation includes four cases, depending on the identities of the buyer who accepts and of the seller who proposes the price:

$$\begin{array}{cc} S_1 & S_2 \\ p_1 & p_2 \\ B_1 & \downarrow \end{array}; \quad \begin{array}{cc} S_1 & S_2 \\ p_1 & p_2 \\ \downarrow & B_1 \end{array}; \quad \begin{array}{cc} S_1 & S_2 \\ p_1 & p_2 \\ B_2 & \downarrow \end{array}; \quad \begin{array}{cc} S_1 & S_2 \\ p_1 & p_2 \\ \downarrow & B_2 \end{array}.$$

Given the symmetry of the traders on the supply side, the first and the last two figures are completely equivalent.

In the present Class, only one buyer trade immediately with a seller at the proposed price, while the other buyer enters, in the following period, a bilateral negotiation with the remaining seller. In our model the latter is just a bilateral bargaining à la Rubinstein with random selection of the proposer at every period. Hence, both the remaining buyer and the unmatched seller expect from the bilateral negotiation one-half of the possible surplus to be divided: $\frac{\lambda}{2}$ if the weak buyer, $\frac{1}{2}$ if the strong buyer is involved in that match.

Consider, for instance, the first case where, in the response phase, buyer $B_1$ accepts the price proposed by seller $S_1$, while buyer $B_2$ rejects both the price offers. The resulting allocation of the goods would be that $B_1$ buys from $S_1$ at price $p_1$, while the low-valuation buyer would trade in a bilateral negotiation with $S_2$ after some delay. Thus, both $B_2$ and $S_2$ each expects a discounted payoff of $\frac{\delta\lambda}{2}$ from the following bilateral bargaining. The resulting expected payoffs from such an allocation would then be $V(S_1) = p_1$, $V(B_1) = 1 - p_1$, $V(S_2) = V(B_2) = \delta\frac{\lambda}{2}$.

Also note that the third and the fourth cases represented in the figure, in which is the low-valuation buyer to accept one price, intuitively can never constitute an equilibrium. In fact, it must be always the case that, if a proposed price is accepted by the buyer with the lowest valuation, it should be accepted also by the highest valuation buyer. The last two cases represented in the figure indeed do not make much sense.

However, it still may well be the case that in a PSSPE equilibrium only the strong buyer accepts a price. In the following we argue that neither this case can be a PSSPE equilibrium.

**Claim 2** There is no PSSPE equilibrium in the negotiation stage where only one buyer accepts a price offer while the remaining buyer rejects both price offers.

**Proof.** In the Appendix ■

The Third Class of PSSPE emerges when, in the response phase, both buyers accept the price offer from the same seller. This situation includes two perfectly symmetric cases, of which we will only consider the first:

$$\begin{array}{cc} S_1 & S_2 \\ p_1 & p_2 \\ B_1, B_2 & ? \end{array} \; ; \; \begin{array}{cc} S_1 & S_2 \\ p_1 & p_2 \\ ? & B_1, B_2 \end{array} .$$

In such a case, one of the two buyers would be randomly selected to trade with $S_1$, while the other will be matched in the next period with the remaining seller, starting a bilateral negotiation. Thus, the payoff of the traders would be as follows: $V(S_1) = p_1$, $V(B_1) = \frac{1}{2}(1 - p_1) + \frac{1}{2}\left(\frac{\delta}{2}\right)$, $V(S_2) = \frac{\delta}{2}\left(\frac{1+\lambda}{2}\right)$, $V(B_2) = \frac{1}{2}(\lambda - p_1) + \frac{1}{2}\left(\frac{\delta\lambda}{2}\right)$.

Of course, also this Class can never constitute a PSSPE equilibrium. The intuition is clearly that both sellers has an incentive to serve only the strong buyer.

**Claim 3** There is no PSSPE equilibrium in the negotiation stage where both buyers accept the same price offer from the same seller.

**Proof.** In the Appendix. ■

In the Fourth Class, both Buyers accept prices from different sellers. Having eliminated all the other seven possible cases, the set of the potential candidate to PSSPE has drastically shrunk just to two remaining cases,

$$\begin{array}{cc} S_1 & S_2 \\ p_1 & p_2 \\ B_1 & B_2 \end{array} \; ; \; \begin{array}{cc} S_1 & S_2 \\ p_1 & p_2 \\ B_2 & B_1 \end{array} .$$

Again we refers to the first case in the figure, being the treatment for the other completely symmetric.

In such a case, all the goods would be immediately sold and each buyers would trade with a different seller, possibly at different prices. The payoffs of the traders would be as follows: $V(S_1) = p_1$, $V(B_1) = 1 - p_1$, $V(S_2) = p_2$, $V(B_2) = \lambda - p_2$.

Notice that, if this allocation was a subgame perfect equilibrium, it would be the case that the following conditions were satisfied.

First, each buyer should expect an higher payoff by accepting his price rather than rejecting it: in the latter case, given that the second buyer is accepting the offer from the other seller, the rejecting buyer would be matched in the next period with the same seller, starting a bilateral negotiation. Thus, clearly it must hold that $1 - p_1 \geq \frac{\delta}{2}$ and that $\lambda - p_2 \geq \delta\frac{\lambda}{2}$. These imply the following conditions on the prices $p_1 \leq 1 - \frac{\delta}{2}$ and $p_2 \leq \lambda\left(1 - \frac{\delta}{2}\right)$, which in turn imply that $p_2 < 1 - \frac{\delta}{2}$.

Second, if this was an equilibrium, it must be true that each buyer would expect an higher payoff by accepting his price rather than the price that the other buyer also accepts, in the latter case being randomly selected to buy the good from that seller only with probability $\frac{1}{2}$, while with the same probability going to bilateral negotiation with the former seller. That is, also the following conditions must hold: $1 - p_1 \geq \frac{1}{2}(1 - p_2) + \frac{1}{2}\left(\frac{\delta}{2}\right)$ and $\lambda - p_2 \geq \frac{1}{2}(\lambda - p_1) + \frac{1}{2}\left(\delta\frac{\lambda}{2}\right)$. These imply the further restrictions on the prices $p_1 \leq \frac{1}{2}\left[\left(1 - \frac{\delta}{2}\right) + p_2\right]$ and $p_2 \leq \frac{1}{2}\left[\lambda\left(1 - \frac{\delta}{2}\right) + p_1\right]$, which in turn, together with $p_2 < 1 - \frac{\delta}{2}$, imply that also $p_1 < 1 - \frac{\delta}{2}$.

This first set of conditions importantly restrict the characteristics of the potential equilibrium, as they rule out the possibility that both sellers symmetrically propose the price that makes the high-valuation buyer as good as in a bilateral negotiation. In other words, if this was the equilibrium, the high-valuation buyer always would gain an higher surplus than in a bilateral negotiation with one single seller, thus benefiting from the presence of the low-valuation buyer in the thin market.

This is not surprising, however, since the case in which both sellers set up prices so high to cut off the low-valuation buyer and to extract the same surplus from the high-valuation buyer as in a bilateral bargaining, can never be an equilibrium because of the competition among the sellers, and of their incentives to undercut, as illustrated in Claim 2.

We prove the following.

Claim 4 There are solutions of an equilibrium selection problem such that there exist two PSSPE equilibria in the $S$-games of the negotiation stage in which any buyer accepts offers from a different seller. In such PSSPE both the sellers propose an identical price $p^* = \lambda\left(1 - \frac{\delta}{2}\right)$. In one

PSSPE the strong buyer accepts the offer from $S_1$, while the weak buyer accepts the one from $S_2$, in the other viceversa.

Proof. In the Appendix. ■

Our result provides conditions under which a PSSPE equilibrium exists in the $S$-games in the negotiation stage where all the trades occur immediately at the same price.

However, we are well aware of the fact that, besides this pure strategies equilibrium in the price-offers phase, there certainly exists a, more challenging, mixed strategies equilibrium in which both sellers propose $p^* = \lambda\left(1 - \frac{\delta}{2}\right)$ with some probability and randomize over the all set of higher prices, in the attempt to cream-skim the highest-valuation demand. To the explicit description of such mixed strategies is, in fact, devoted further research.

### 3.3 $B$-games

We then consider all the subgames of the original game starting with the selection of the buyers as proposers.

Once again, thanks to the hypothesis of stationarity, working out the candidate PSSPE in the $B$-games, we just need to consider any generic round of the negotiation stage in which the buyers have been selected to make offers. Within that generic round, using a standard backward induction argument, we first analyze the acceptance game played by the sellers in the response phase, and then, the proposal game played by the buyers in the price-offers phase.

Therefore, denote $p_1$ and $p_2$ the prices offered simultaneously and independently by sellers $B_1$ and $B_2$, respectively, in the price-offers phase of a generic $B$-game.

Again, we may easily characterize the acceptance or rejection game played by the two sellers in the price-response phase. In fact, the strategies available to each of the responding sellers are just three: either "Accept $p_1$", or "Accept $p_2$", or, finally "Reject both prices". Hence, the normal-form representation in Figure 2 describes the response game:

By comparing the resulting payoffs and after some manipulations, we can work out a full description of the best response strategies by any seller in the response phase.

In fact, consider first the seller $S_1$. If the other seller plays strategy "Accept $p_1$", then the best response by $S_1$ is to also "Accept $p_1$" if

$$\begin{cases} p_1 \geq \frac{\delta\lambda}{2} \\ p_2 \leq \frac{1}{2}p_1 + \frac{\delta\lambda}{4} \end{cases},$$

|  |  | **S₂** |  |  |
|---|---|---|---|---|
|  |  | ***p₁*** | ***p₂*** | **ø** |
| **S₁** | ***p₁*** | $\frac{1}{2}p_1 + \frac{dl}{4}, \frac{1}{2}p_1 + \frac{dl}{4}$ | $p_1 , p_2$ | $p_1 , \frac{dl}{2}$ |
|  | ***p₂*** | $p_2, \; p_1$ | $\frac{1}{2}p_2 + \frac{d}{4}, \frac{1}{2}p_2 + \frac{d}{4}$ | $p_2 , \frac{d}{2}$ |
|  | **ø** | $\frac{dl}{2}, \; p_1$ | $\frac{d}{2}, \; p_2$ | $dV(S_1), dV(S_2)$ |

Figure 2:

or, alternatively, to choose to "Accept $p_2$" if

$$\begin{cases} p_2 \geq \frac{\delta\lambda}{2} \\ p_2 \geq \frac{1}{2}p_1 + \frac{\delta\lambda}{4} \end{cases},$$

or, finally, to "Reject both prices" if

$$\begin{cases} p_1 \leq \frac{\delta\lambda}{2} \\ p_2 \leq \frac{\delta\lambda}{2} \end{cases}.$$

On the other hand, given that the other seller $S_2$ is adopting the strategy to "Accept $p_2$", the best response by seller $S_1$ is to opt for "Accept $p_1$" if

$$\begin{cases} p_1 \geq \frac{\delta}{2} \\ p_1 \geq \frac{1}{2}p_2 + \frac{\delta}{4} \end{cases},$$

or, to also choose to "Accept $p_2$", if

$$\begin{cases} p_2 \geq \frac{\delta}{2} \\ p_1 \leq \frac{1}{2}p_2 + \frac{\delta}{4} \end{cases},$$

or, again, to reject both price offers if

$$\begin{cases} p_1 \leq \frac{\delta}{2} \\ p_2 \leq \frac{\delta}{2} \end{cases}.$$

Finally, given that the other seller $S_2$ choose to "Reject both prices", the best response for $S_1$ is to decide to accept price $p_1$ if

$$\begin{cases} p_1 \geq p_2 \\ p_1 \geq \delta V(S_1) \end{cases},$$

or, alternatively, to opt for "Accept $p_2$" if

$$\begin{cases} p_2 \geq p_1 \\ p_2 \geq \delta V(S_1) \end{cases},$$

or, finally, to also decide to reject both offers if

$$\begin{cases} p_1 \leq \delta V(S_1) \\ p_2 \leq \delta V(S_1) \end{cases},$$

where, again, the discounted value of the expected payoff to enter a new round of the negotiation stage when all the four traders are still active in the market.

Given the symmetry of the traders on the supply side of the thin market, the description of the set of best responses for seller $S_2$ is perfectly equivalent to the one drawn for $S_1$.

Hence, it is possible to derive under which conditions a given pair of strategies by the sellers may possibly represent a possible Nash equilibrium of the acceptance game in the response phase. However, once again, the sets of symmetric restrictions over the $(p_1, p_2)$ space turn out to be rather inconclusive in providing any useful conditions.

Therefore, we should once again consider the overall negotiation process occurring in a generic round of a $B$-game. In fact, some of the potential Nash equilibria in the response phase turn out to be impossible to substain simply because there are no equilibria in the price-offers phase that can possibly generate such strategies in any of their subgames.

As before, an helpful starting point is to characterize all the possible PSSPE equilibria emerging in a $B$-game. Analogously to the analysis of $S$-games, exactly 9 possible equilibrium allocations of the goods may emerge from a bargaining period in which the buyers make proposals.

We classify again the 9 possible allocations in 4 classes and we show that some of them can never represent a subgame perfect equilibrium of the $B$-games. We thus gradually restrict the set of the potential equilibria to fewer classes by sequentially discarding some cases.

Finally, we show that in the $B$-games there exist two possible PSSPE allocations according to the values of the impatience rate $\delta$ and of the heterogeneity factor $\lambda$. In particular, there always exists a PSSPE where trade happens with delay, while for mild heterogeneity and high impatience there exists an alternative PSSPE where all the potential surplus from the exchange is exploited immediately. However, in both the PSSPE, trading involves coexistence of different prices.

The Fifth Class emerges where, in the response phase, both sellers reject any of the offers $p_1$ and $p_2$ proposed by the two buyers. We may represent the candidate equilibrium by the figure

$$\begin{array}{cc} B_1 & B_2 \\ p_1 & p_2 \\ ? & ? \end{array}$$

where the last row indicates, as usual, the set of the sellers accepting the price $p_i$ by the buyer $B_i$, $i = 1, 2$.

In such a case all the players do not trade and enter the next round, with a new selection of the proposers. Their relative payoffs are then given by the discounted value of the expected payoff by entering a new stage of negotiation. It is then immediate to state the following.

**Claim 5** There is no PSSPE equilibrium in the negotiation stage where both sellers reject both the prices offered by the buyers.

*Proof.* In the Appendix. ■

The Sixth Class embodies situations when only one from the two sellers accepts one price, while the other rejects both. This class includes four cases, depending on the identities of the seller who accepts and of the buyer who proposes the price:

$$\begin{array}{cc} B_1 & B_2 \\ p_1 & p_2 \\ S_1 & ? \end{array} ; \begin{array}{cc} B_1 & B_2 \\ p_1 & p_2 \\ ? & S_1 \end{array} ; \begin{array}{cc} B_1 & B_2 \\ p_1 & p_2 \\ S_2 & ? \end{array} ; \begin{array}{cc} B_1 & B_2 \\ p_1 & p_2 \\ ? & S_2 \end{array} .$$

Given the symmetry of the game, we only consider the allocations as represented by the first and the second figures, then adapting the findings to the other identical seller.

In such cases, only one seller trade immediately with a buyer at the proposed price, while the other seller enters, in the following period, a new bilateral negotiation with the remaining buyer. We model the latter negotiation as a Rubinstein bilateral bargaining with random selection of the proposer at every period. Hence, both the remaining seller and the unmatched buyer expect from the bilateral negotiation one-half of the possible surplus to be divided. Thus, in the first case both $B_2$ and $S_2$ each expects a discounted payoff of $\delta\frac{\lambda}{2}$, while in the second case, both $B_1$ and $S_2$ each expects a discounted payoff of $\delta\frac{1}{2}$.

First note that the second and the fourth cases represented in the figure, in which the only accepted price is the one proposed by the low-valuation, intuitively can never be an equilibrium. In fact, it must be always the case that, if the buyer with the lowest valuation can propose a price that will be accepted, this might be a fortiori proposed also by the highest valuation buyer.

The last two cases represented in the figure indeed do not make much sense, since it is clear that the high-valuation buyer can always deviate by proposing a price $p_1^0 = p_2 + \varepsilon$, then attracting the seller that is already accepting $p_2$. Thus these cases can never be equilibria.

Furthermore, note that the perfect symmetry among the sellers raises another major question about allocations within this class: how can it be possible that two identical sellers behave differently, one accepting and the other rejecting the same offer in equilibrium?

In the following we will only refer to the first case in the figure, but our argument clearly extends by symmetry to the third case, and, a fortiori, to the other two.

Consider the case where, as the outcome of the negotiation, buyer $S_1$ accepts the price proposed by seller $B_1$, while buyer $B_2$ rejects both the prices offered by the two sellers.

The resulting allocation of the goods would be that $S_1$ sells to $B_1$ at price $p_1$, while the other seller would trade in a bilateral negotiation with $B_2$ after some delay. The resulting expected payoffs from such an allocation would be $V(S_1) = p_1$, $V(B_1) = 1 - p_1$, $V(S_2) = V(B_2) = \delta\frac{\lambda}{2}$.

We show the following.

**Claim 6** There is no PSSPE equilibrium in the negotiation stage where only one seller accepts a price offer while the other rejects both proposals from the buyers.

**Proof.** In the Appendix. ■

Having eliminated all the previous Classes of possible allocations in the $B$-games, the set of the potential candidate to subgame perfect equilibria has drastically restricted to just two remaining Classes.

The *Seventh Class* embraces all the situations where both buyers accepts the price offer from the same seller. This situation includes two asymmetric cases, that should better be treated separately:

$$\begin{array}{ll} B_1 & B_2 \\ p_1 & p_2 \\ S_1, S_2 & ? \end{array} \quad ; \quad \begin{array}{ll} B_1 & B_2 \\ p_1 & p_2 \\ ? & S_1, S_2 \end{array} .$$

In the first case, both sellers accept the offer by the high-valuation buyer. In such a case, one of the sellers would be randomly selected to trade with $B_1$, while the other will be matched in the next period with the remaining buyer, starting a bilateral negotiation. Thus, the payoff of the traders would be as follows: $V(S_1) = V(S_2) = \frac{1}{2}p_1 + \frac{1}{2}\left(\delta\frac{\lambda}{2}\right)$, $V(B_1) = 1 - p_1$, $V(B_2) = \delta\frac{\lambda}{2}$.

In the second case, both sellers accept instead the offer by the low-valuation buyer. Again, one of the sellers would be randomly selected to trade with $B_2$, while the other will be matched in the next period with the high-valuation buyer, starting a bilateral negotiation. Thus, the payoff of the traders would be as follows: $V(S_1) = V(S_2) = \frac{1}{2}p_1 + \frac{1}{2}\left(\frac{\delta}{2}\right)$, $V(B_1) = \frac{\delta}{2}$, $V(B_2) = \lambda - p_2$.

As concerns the second case, clearly any price that is offered by the weak buyer may also be afforded by the strong buyer as well, so that the latter may always undermine the present allocation by attracting at least one seller, thus gaining a surplus certainly greater than in a bilateral negotiation. In fact, the following holds.

**Claim 7** There is no PSSPE equilibrium in the negotiation stage where both sellers accept the price offered by the weak buyer.

*Proof.* In the Appendix. ■

Concerning the first case, denote $p_1^*$ and $p_2^*$ the prices offered by $B_1$ and $B_2$ respectively. Then, we prove the following Proposition.

**Proposition 8** For any value of $\delta < 1$, there always exists one PSSPE equilibrium of the negotiation stage where both sellers, in the response phase, accept the price offered by the strong buyer. In particular,

- When $\lambda < \frac{\delta}{4-3\delta}$, there exists a PSSPE equilibrium where the proposals by the strong and the weak buyer in the price offer phase are, respectively,

$$\left\{ \begin{array}{c} p_1^* \text{ s.t. } \frac{1}{2}\left(p_1^* + \frac{\delta\lambda}{2}\right) = \lambda\left(1 - \frac{\delta}{2}\right) \\ p_2^* = \lambda\left(1 - \frac{\delta}{2}\right) \end{array} \right. ,$$

  and where both sellers accept $p_1^*$.

- When $\lambda > \frac{\delta}{4-3\delta}$, there exists a a PSSPE equilibrium where the proposals by the strong and the weak buyer in the price offer phase are, respectively,

$$\left\{ \begin{array}{c} p_1^* = \frac{\delta}{2} \\ p_2^* = \frac{\delta}{4}(1+\lambda) \end{array} \right. ,$$

  and where both sellers accept $p_1^*$.

*Proof.* In the Appendix. ■

The last Class in the $B$-games embraces two symmetric situations where both sellers accept prices from different buyers,

$$\begin{array}{cc} B_1 & B_2 \\ p_1 & p_2 \\ S_1 & S_2 \end{array} \; ; \; \begin{array}{cc} B_1 & B_2 \\ p_1 & p_2 \\ S_2 & S_1 \end{array} :$$

again we refers to the first case in the figure, being the treatment for the other completely symmetric.

In such a case, all the goods would be immediately sold and each buyer would trade with a different seller, possibly at different prices. The payoffs of the traders would be as follows: $V(S_1) = p_1$, $V(B_1) = 1 - p_1$, $V(S_2) = p_2$, $V(B_2) = \lambda - p_2$.

Notice that, if this allocation is a PSSPE equilibrium, it would be the case that the following conditions were satisfied.

First, each seller should expect an higher payoff by accepting his price rather than rejecting it: in the latter case, given that the second seller is accepting the offer from the other buyer, the rejecting seller would be matched in the next period with the same buyer, starting a bilateral negotiation. Thus, clearly it must hold that $p_1 \geq \frac{\delta}{2}$ and that $p_2 \geq \delta\frac{\lambda}{2}$. These imply that $p_1 > \delta\frac{\lambda}{2}$.

Second, if this was an equilibrium, it must be true that each seller would expect an higher payoff by accepting her price rather than the price that the other seller also accepts, in the latter case being randomly selected to sell the good to that buyer only with probability $\frac{1}{2}$, while with the same probability going to bilateral negotiation with the former buyer. That is, also the following conditions must hold: $p_1 \geq \frac{1}{2}p_2 + \frac{1}{2}\left(\frac{\delta}{2}\right)$ and $p_2 \geq \frac{1}{2}p_1 + \frac{1}{2}\left(\delta\frac{\lambda}{2}\right)$. These two inequalities, together with $\lambda < 1$ and the previous $p_1 \geq \frac{\delta}{2}$ and $p_2 \geq \delta\frac{\lambda}{2}$ state the further conditions to hold for this allocation being an equilibrium:

$$\begin{cases} p_2 - \frac{1}{2}p_1 \geq \delta\frac{\lambda}{4} \\ p_1 > \delta\frac{\lambda}{2} \end{cases}$$

which in turn also imply that $p_2 > \delta\frac{\lambda}{2}$. That is, if this was an equilibrium, it would imply that the price offered by $B_2$ would be higher than the profit $S_2$ can get by going to bilateral negotiation with him: the weak buyer would allow the seller which he trades with to gain an extra-profit with respect to her outside option.

Moreover, intuitively it seems reasonable to believe that the price that may be substained in equilibrium by the weak buyer would never exceed the one proposed by the strong buyer, that is $p_2 \leq p_1$.

Putting together all the above conditions to hold for this allocation being an equilibrium, the following system it turns out

$$\begin{cases} p_1 \geq \frac{\delta}{2} \\ p_2 > \delta\frac{\lambda}{2} \\ p_2 \geq \frac{1}{2}p_1 + \delta\frac{\lambda}{4} \end{cases}.$$

Clearly, it follows that the only possible equilibrium strategy by strong buyer is the price offer $p_1 = \frac{\delta}{2}$, which in turn implies the last two conditions may be written as $p_2 \geq \delta\frac{\lambda}{4} + \frac{\delta}{4} > \delta\frac{\lambda}{2}$. This, finally, gives us the values for the buyers' proposals to be an equilibrium,

$$\begin{cases} p_1 = \frac{\delta}{2} \\ p_2 = \frac{\delta}{4}(1+\lambda) \end{cases}$$

as setting any price offer higher than the latters would clearly be a dominated strategy for the buyers. Notice that, as $\lambda < 1$, it always holds that $p_1 > p_2 > \frac{\delta\lambda}{2}$.

We prove the following.

**Proposition 9** For $\lambda > \frac{\delta}{4-3\delta}$, there exist a PSSPE equilibrium of the negotiation stage where, in the price-offers phase the strong and the weak buyer, propose respectively,

$$\begin{cases} p_1^* = \frac{\delta}{2} \\ p_2^* = \frac{\delta}{4}(1+\lambda) \end{cases},$$

and, in the response phase, the sellers accept the price offers from different buyers.

*Proof.* In the Appendix. ■

Thus, depending on the levels of the intertemporal discount rate and of the index of heterogeneity in the reservation prices, the $B$-subgames show two alternative PSSPE allocations. Notice that in both equilibria, the buyers end up paying different prices.

## 3.4 Complete Network: a Summary of the Results

Thus we have attempted to show that the described bargaining game among two identical sellers and two heterogeneous buyers given a complete bipartite graph exhibits two alternative pure and stationary strategies subgame perfect equilibrium (PSSPE) - depending on the original configuration of the traders characteristics - whenever the buyers are selected to make an offer ($B$-games) and one subgame perfect equilibrium in pure strategies when the sellers are selected to make an offer ($S$-games).

In particular, in both the PSSPE in the $B$-games, two different prices emerge in the endogenous process of negotiations. This is rather surprising to occur in such a thin market, specially in the case where trade occurs at the same time.

Furthermore, in one PSSPE both sellers accept immediately the price offered by the high-valuation buyer. Because of the random selection of the seller entitled to trade with the high-valuation buyer, one seller and the low-valuation buyer trade only with delays at a different price. Thus both different prices and inefficiency due to costly delays emerge in the equilibrium.

It seems then interesting to check which outcome may emerge from negotiations in case agents bargain given a less connected network. That is, we should now address our attention to network structures where only a link has been severed with respect to the complete bipartite graph. The question is: may they substain alternative, possibly more efficient, equilibrium allocations?

## 3.5 Asymmetrically Connected Networks

### 3.5.1 The Strong Buyer Having Two Links

### 3.5.2 The Weak Buyer Having Two Links

## 3.6 Disconnected Networks

To be written

# 4 The Network Formation Game

To be written

# 5 Conclusions

To be written

# 6 Appendix. Proofs

## 6.1 Proof of Claim 1

First, note that if this was indeed the sellers' equilibrium stationary strategy, must be the case that in any period, when they are selected to make offers, both the sellers keep on proposing prices so high that both buyers reject them.

This means that the offered prices make both buyers worse off than their continuation payoffs. That is, the followings must hold: $p_i > 1 - \delta W(B_1)$ and $p_i > \lambda - \delta W(B_2)$, with $i = 1, 2$ and with $W(B_j)$ being the expected payoff by buyer $j = 1, 2$ by rejecting the offers and entering a new round.

Consider now the lower bound of $W(B_1)$, that is the minimum surplus buyer $B_1$ may expect from trade by entering a new round of negotiations.

If the one described above is indeed a stationary equilibrium, at every period with probability $\frac{1}{2}$ the sellers' offers are rejected and all the traders go further with the negotiation. Alternatively, again with probability $\frac{1}{2}$, the buyers are selected to make offers.

In such a case, it is easy to see that the minimum payoff buyer $B_1$ may obtain by proposing a price is $1 - \lambda - \varepsilon$, with $\varepsilon$ infinitesimally small. In fact, $B_1$ can always get at least that surplus by proposing a price $\lambda + \varepsilon$: such a price is immediately accepted by both sellers since not only it is above the highest price possibly offered by $B_2$, but also, as $\lambda > \frac{1}{2}$, it is strictly higher than what sellers can get in bilateral negotiation from any of the buyer. On the other hand, note that, of course, such a price makes $B_1$ worse off than in a bilateral negotiation with a single seller.

Thus define $W_{\min}(B_1)$ as the lowest continuation payoff buyer $B_1$ may expect if the above case was indeed a subgame perfect equilibrium: having shown that $W_{\min}(B_1) = \frac{1}{2}\delta W_{\min}(B_1) + \frac{1}{2}(1 - \lambda - \varepsilon)$ gives $W_{\min}(B_1) = \frac{1-\lambda-\varepsilon}{2-\delta}$.

Then, for the above condition $p_i > 1 - \delta W(B_1)$ with $i = 1, 2$ holding in equilibrium, must necessarily be that $p_i > 1 - \delta W_{\min}(B_1) = 1 + \frac{\delta(\lambda+\varepsilon)}{2-\delta}$.

Consider now the sellers. As they propose offers that are rejected by both buyers, their expected payoffs equal $\delta W(S_i)$, where $W(S_i)$ is the expected payoff by seller $i = 1, 2$ by entering a bargaining round before a new selection of the proposer.

Define $W_{\max}(S_i)$ as the highest payoff each seller $i = 1, 2$ may expect from a new bargaining period if their above strategies were indeed a subgame perfect equilibrium. These are necessarily associated to the lowest expected payoffs by the buyers when the latter would be selected to make an offer. That is, the maximum the sellers may obtain by keep on proposing offers that will be rejected by both buyers needs to correspond to a price $\lambda + \varepsilon$ proposed by $B_1$ when the buyers are selected to make an offer.

In the latter case, it must be that in a subgame perfect equilibrium both sellers would accept the offered price $\lambda + \varepsilon$, since a rejection will clearly give a lower payoff. In fact, if, say, $S_1$ rejected the price $\lambda + \varepsilon$, the best it might happen is that $S_2$ also rejected that price, which gives at most a payoff of $\delta W_{\max}(S_i)$, that is by definition lower than what she would get accepting it. However, if $S_2$ will accept the price $\lambda + \varepsilon$, $S_1$ can obtain only $\delta\frac{\lambda}{2}$ in the subsequent bilateral negotiation with $B_2$.

As both sellers in equilibrium will accept the same price $\lambda + \varepsilon$ by $B_1$, a random selection of a winner will be in order to solve the tie in the allocation: one of the two seller will be chosen to buy from $B_1$ at price $\lambda + \varepsilon$, while the other will enter a bilateral negotiation with $B_2$.

Thus, if their above strategies were indeed a subgame perfect equilibrium, the symmetric highest payoff each seller may expect from a new bargaining round is $W_{\max}(S_i) = \frac{1}{2}\delta W_{\max}(S_i) + \frac{1}{2}\left[\frac{\lambda+\varepsilon}{2} + \frac{\delta}{2}\left(\frac{\lambda}{2}\right)\right]$, that is $W_{\max}(S_i) = \frac{\lambda}{4}\left(\frac{2+\delta}{2-\delta}\right) + \frac{\varepsilon}{2(2-\delta)}$.

Hence, if the sellers adopted the above strategies such that the equilibrium offered prices are never accepted, that is if $p_i > 1 + \frac{\delta(\lambda+\varepsilon)}{2-\delta}$, then they would obtain at most an expected payoff $\delta W_{\max}(S_i) = \delta \left[\frac{\lambda}{4}\left(\frac{2+\delta}{2-\delta}\right) + \frac{\varepsilon}{2(2-\delta)}\right]$. But, then, it is easy to verify that the latter strategies can not be an equilibrium for traders not too impatient, that is for $\delta$ sufficiently high.

In fact, let one of the seller, say $S_1$, to deviate by proposing, for instance, a price $p_1$ exactly equal to $1 + \frac{\delta(\lambda+\varepsilon)}{2-\delta}$. This is the lowest price that leaves the high-valuation buyer indifferent between accepting and rejecting an offer. By comparison, it is immediately checked that this strategy makes $S_1$ better off with respect to the one of proposing unacceptable offers: in fact, for values of $\delta$ sufficiently high and for $\varepsilon$ small enough, it always holds that $\lambda > \frac{2\delta - \delta\varepsilon - 4}{\delta(1-\frac{\delta}{2})}$, that is, offering $p_1$ makes $S_1$'s profit strictly bigger than $W_{\max}(S_1)$, the maximum payoff she may expect with the latter strategy. Furthermore, by analogy it may be shown that, even proposing a price so low that it may attract the lowest-valuation buyer, one of the seller may benefit by deviating from the described strategy, at least for large ranges of the relevant primitive parameters. Thus, the described strategies can never be a subgame perfect equilibrium for values of $\delta$ sufficiently high.

## 6.2 Proof of Claim 2

The proof here reported refers to the first case in the figure, but it clearly extends by symmetry to the second case, and, a fortiori, to the other two.

Notice that, if this allocation was a PSSPE equilibrium, it would be the case that the following conditions describing such an equilibrium in the response phase were satisfied.

First, it must be the case that $p_2 \geq p_1$, for otherwise $B_1$ had accepted the lower price $p_2$ instead.

Second, for the price $p_1$ to be accepted by buyer $B_1$ it must be set to a level such that the latter is indifferent between accepting it, gaining $1 - p_1$, and rejecting it going to a further bargaining period in a situation such as the one described in the First Class.

Third, must be the case that, by rejecting both offers, buyer $B_2$ expects an higher payoff than by accepting one of the two. Therefore, it must be that $\lambda - p_2 < \frac{\delta\lambda}{2}$, that is $p_2 > \lambda\left(1 - \frac{\delta}{2}\right)$.

Furthermore, if buyer $B_2$ accepted the same price $p_1$, he would be randomly selected with probability $\frac{1}{2}$ to keep the good rather than going to bilateral negotiations with the remaining seller. Then if this was an equilibrium it must be that the expected payoff for $B_2$ by accepting $p_1$ would never be lower than the payoff he may obtain by rejecting and going directly to bilateral negotiation, that is the following must hold:

$\frac{1}{2}(\lambda - p_1) + \frac{1}{2}\left(\delta\frac{\lambda}{2}\right) < \frac{\delta\lambda}{2}$.

The latter implies that, if this Class was a PSSPE, the emerging prices would be such that $p_2 \geq p_1 > \lambda\left(1 - \frac{\delta}{2}\right)$.

We have then described the conditions to hold in the acceptance game if such a Class was a Nash equilibrium in the response phase. However, it is immediate to see that these conditions can not be supported by any equilibrium in the price offers game by the sellers.

In fact, consider a deviation by $S_2$ from the described strategy to propose $p_2 > \lambda\left(1 - \frac{\delta}{2}\right)$. For instance, she may deviate by proposing a price $p_2' = p_1 - \varepsilon \geq \lambda\left(1 - \frac{\delta}{2}\right)$, which turns out to always be a profitable deviation.

In fact, if she deviates by proposing $p_2' = p_1 - \varepsilon > \lambda\left(1 - \frac{\delta}{2}\right)$, while the weak buyer will keep on rejecting both offers, the strong buyer will immediately accept $p_2'$. Moreover, even if the only possible deviation is $p_2' = p_1 - \varepsilon = \lambda\left(1 - \frac{\delta}{2}\right)$, it can be easily checked using the above payoff matrix that different equilibria can arise in the generated response game. Either the strong buyer accepts $p_2'$ while the weak buyer rejects both offers, or, depending on the initial value of $p_1$, either both buyers accept $p_2'$ or, alternatively, the weak buyer accepts it while the strong buyer keeps on accepting $p_1$. In all these equilibria, however, $S_2$ will surely sell the good to one of the buyers being able to earn $p_1 - \varepsilon \geq \lambda\left(1 - \frac{\delta}{2}\right)$ rather than $\frac{\delta\lambda}{2}$. But, as it always holds that $\lambda\left(1 - \frac{\delta}{2}\right) > \frac{\delta\lambda}{2}$, such a deviation is indeed profitable.

But this contradicts the assumption that the present allocation is a PSSPE equilibrium. For the other three cases in the present Class, the same logic applies.

## 6.3 Proof of Claim 3

Notice that, if there is a PSSPE in the present Class, it needs to be the case that the following conditions for the response phase would be satisfied.

First, as usual it must be the case that $p_2 \geq p_1$, for otherwise at least one buyer would have accepted the lower price $p_2$ instead.

Second, it must hold that the expected payoff for $B_1$ from accepting the same price which also $B_2$ is accepting is higher than the expected surplus attached to any of the possible alternatives.

In particular, if this was an equilibrium the expected payoff by accepting $p_1$ given that also $B_2$ accepts $p_1$, must be at least equal to the one in case $B_1$ accepted $p_2$ instead, given that $B_2$ accepts $p_1$, that is $\frac{1}{2}\left(1 + \frac{\delta}{2}\right) - \frac{1}{2}p_1 \geq 1 - p_2$.

Analogously, the expected surplus by accepting $p_1$ given that also $B_2$ accepts $p_1$, must be at least equal to the one in case $B_1$ rejected both offer, given that $B_2$ accepts $p_1$. In the latter case, $B_1$ would be matched in the next period with the remaining seller, starting a bilateral negotiation: thus, if this was an equilibrium, it would be true that is $\frac{1}{2}\left(1+\frac{\delta}{2}\right)-\frac{1}{2}p_1 \geq \frac{\delta}{2}$.

Manipulating these inequalities gives the set of conditions for $B_1$'s strategy being an equilibrium,

$$\begin{cases} p_1 \leq 1-\frac{\delta}{2} \\ p_2 \geq \frac{1}{2}p_1+\frac{1}{2}\left(1-\frac{\delta}{2}\right) \end{cases},$$

which, together, also prove formally the first described condition, as $p_1 \leq \frac{1}{2}p_1+\frac{1}{2}\left(1-\frac{\delta}{2}\right) \leq p_2$.

The same logic leads to the set of analogous conditions for $B_2$'s strategy being an equilibrium in the response phase:

$$\begin{cases} p_1 \leq \lambda\left(1-\frac{\delta}{2}\right) \\ p_2 \geq \frac{1}{2}p_1+\frac{1}{2}\lambda\left(1-\frac{\delta}{2}\right) \end{cases}.$$

Now, notice that, as $\lambda < 1$, the first condition for $B_2$ implies clearly the strict inequality of the analogous condition for $B_1$, that is $p_1 < 1-\frac{\delta}{2}$.

This in turn allows to further restrict the first condition on the relative size of the two prices. In fact, note that, if this allocation was an equilibrium, it must hold the strict inequality $p_2 > p_1$. To see why, assume at the contrary that $p_2 = p_1$. If this was the case it is immediate to show that $B_1$, taking as given the acceptance by $B_2$, would deviate by accepting $p_2$ instead. In fact, by deviating and accepting $p_2$ buyer $B_1$ would obtain $1-p_2$ rather than $\frac{1}{2}\left(1+\frac{\delta}{2}\right)-\frac{1}{2}p_1$. If it was true that $p_2 = p_1$, $B_1$ would deviate if and only if $1-p_1 \geq \frac{1}{2}\left(1+\frac{\delta}{2}\right)-\frac{1}{2}p_1$, which is always verified as we have found that in equilibrium must hold that $p_1 < 1-\frac{\delta}{2}$. Thus if there is a PSSPE in this Class, it must also hold that $p_2 > p_1$.

However, it is then immediate to see that such an equilibrium in the response game can not be supported by any equilibrium in the price-offers game.

In fact, consider the seller $S_1$. In the original allocation he offered $p_1 \leq \lambda\left(1-\frac{\delta}{2}\right) < 1-\frac{\delta}{2}$, that is a price such that the low-valuation buyer is indifferent between accepting it or rejecting it, while the high-valuation buyer is strictly better off accepting it. However, as also $p_1 < p_2$, seller $S_1$ may profitably deviate by proposing a price $p_1' = p_1+\varepsilon$ which is still below $p_2$. In fact, in such a way, she makes a proposal that will be accepted by the high-valuation buyer only, giving her an higher payoff than the initial situation.

## 6.4 Proof of Claim 4

The proof will refer to the PSSPE where the strong buyer accepts the offer from $S_1$, while the weak buyer accepts the one from $S_2$, being the other case absolutely equivalent.

First, notice that, given the two conditions $p_1 < 1 - \frac{\delta}{2}$ and $p_2 \le \lambda\left(1 - \frac{\delta}{2}\right)$ we must look for a candidate equilibrium only in the space strictly below the locus $p_1 = 1 - \frac{\delta}{2}$ corresponding to the price that makes the high-valuation buyer indifferent between buying in the thin market or going to bilateral negotiation. Furthermore, we also must look for a candidate equilibrium where the price charged to the low-valuation buyer is never above the line $p_2 = \lambda\left(1 - \frac{\delta}{2}\right)$, since, to be the equilibrium in this Class, must also be that the price charged to $B_2$ makes him not worse off than in bilateral negotiation.

Hence, we must rule out from the set of equilibrium candidates all the possible situations where both prices are above line $p_2 = \lambda\left(1 - \frac{\delta}{2}\right)$, as these correspond to the cases described in the second Class where both sellers choose to serve only the high-valuation buyer.

Furthermore, it can be argued that if $p_1^*$ and $p_2^*$ are equilibrium price-offers which are accepted in the response game by $B_1$ and $B_2$, respectively, they must necessarily be equal: $p_1 = p_2$.

In fact, suppose at the contrary that in equilibrium the two sellers propose different prices to the buyers, that is $p_1^* \neq p_2^*$. Intuitively the sellers might want to try to extract an higher price from the highest-valuation buyer. Thus, since it does not make much sense that the price charged to the strong buyer would be lower than the one charged to the low-valuation buyer, assuming $p_1^* \neq p_2^*$ it is equivalent to assume $p_1^* > p_2^*$.

Notice that, as the condition $p_2 \le \lambda\left(1 - \frac{\delta}{2}\right)$ must hold if this is an equilibrium, the seller $S_2$ is charging a price that is still accepted by the weak buyer. However, given that $S_1$ is proposing a price $p_1 > p_2$, seller $S_2$ can indeed profitably deviate from her original strategy. In fact, two subcases are in order.

In the first subcase, indeed, $S_2$ sets a price $p_2 < \lambda\left(1 - \frac{\delta}{2}\right)$, while $S_1$ proposes a price $\lambda\left(1 - \frac{\delta}{2}\right) \ge p_1 > p_2$. It is immediate to observe that this subcase can never be a PSSPE equilibrium: $S_2$ can infact profitably deviate by proposing a price $p_2' = p_2 + \varepsilon < p_1$, with $\varepsilon > 0$ small, which is still accepted at least by the low-valuation buyer.

In the second subcase, $S_2$ sets a price $p_2 \le \lambda\left(1 - \frac{\delta}{2}\right)$, while $S_1$ proposes a price $p_1 > \lambda\left(1 - \frac{\delta}{2}\right) \ge p_2$. It is easy to see that neither this subcase can ever be an equilibrium, as $S_2$ again may increase the price offered, by proposing a price $p_2' = p_1 - \varepsilon$, with $\varepsilon > 0$. As long as $p_2' > \lambda\left(1 - \frac{\delta}{2}\right)$, this deviation is profitable for $S_2$: in fact, in the following response game, the weak buyer would prefer to reject both offers, while

$B_1$ would accept the price $p_2^0$. This, in turn, would clearly ensure the deviating $S_2$ a higher payoff than the original strategy $p_2 \le \lambda\left(1 - \frac{\delta}{2}\right)$.

Hence, if it exists an equilibrium in this Class, it would imply that the sellers charge an identical price.

Also note that the case in which both sellers charge an identical price $\lambda\left(1 - \frac{\delta}{2}\right) < p < 1 - \frac{\delta}{2}$ is not compatible with the conditions describing a PSSPE equilibrium in this Class, and, moreover, would never constitute an equilibrium: clearly each seller, in order to capture the demand from the strong buyer, would have a sharp incentive to deviate by proposing a price strictly below the one charged by the competitor. Not surprisingly, this situation may be seen as a Bertrand duopoly competing by undercutting for serving a single-customer demand.

Hence, only two possible cases remain to be analyzed as candidate to the equilibrium: either both sellers charge some identical price $p < \lambda\left(1 - \frac{\delta}{2}\right)$ or both set a price $p = \lambda\left(1 - \frac{\delta}{2}\right)$.

However, once again, intuition suggests neither the former case can ever constitute an equilibrium. In fact, the seller serving the weak buyer would always prefer to charge a slightly higher price, anticipating that, in the subsequent response phase, such a deviation would always be accepted at least by the weak buyer: in fact, either by rejecting all the proposals or by accepting the other offer and by bearing the risk of a random matching, $B_2$ would always be worse off.

Therefore, one can argue that, if a PSSPE exists, then it implies that both sellers charge some identical price $p^* = \lambda\left(1 - \frac{\delta}{2}\right)$.

In particular, consider a situation where in the price-offers phase, both $S_1$ and $S_2$ propose $p_1^* = p_2^* = \lambda\left(1 - \frac{\delta}{2}\right)$ and, in the response phase, the strong buyer accepts $p_1$ while $B_2$ accepts $p_2$. Now we formally prove that this is indeed a PSSPE of the negotiation stage.

In fact, by substituting the values for the prices $p^*$ in the payoffs of the acceptance game we obtain the normal form matrix of the response phase in Figure 3:

As it always holds that $1 - \lambda > \frac{\delta}{2}(1 - \lambda)$, implying both that $1 - \lambda + \frac{\delta\lambda}{2} > \frac{1-\lambda}{2} + \frac{\delta}{4}(1 + \lambda)$ and that $1 - \lambda + \frac{\delta\lambda}{2} > \frac{\delta}{2}$, it is immediate to see that the response game has at least two pure strategies (asymmetric) Nash equilibria, where both buyers accept a price; either $B_1$ accepts $p_1$ while $B_2$ accepts $p_2$ or viceversa.

Furthermore, it is reasonable to believe that the continuation values of the buyers show some upper bound. In particular, we assume that the buyers' expected payoff from entering a new round of the negotiation stage satisfy the two following restrictions

$$\begin{cases} \delta W(B_1) \le 1 - \lambda\left(1 - \frac{\delta}{2}\right) \\ \delta W(B_2) \le \frac{\delta\lambda}{2} \end{cases}.$$

|  |  | $B_2$ |  |  |
|---|---|---|---|---|
|  |  | $p_1^*$ | $p_2^*$ | ø |
| $B_1$ | $p_1^*$ | $\frac{1-l}{2}+\frac{d}{4}(1+l), \frac{dl}{2}$ | $1-l\left(1-\frac{d}{2}\right), \frac{dl}{2}$ | $1-l\left(1-\frac{d}{2}\right), \frac{dl}{2}$ |
|  | $p_2^*$ | $1-l\left(1-\frac{d}{2}\right), \frac{dl}{2}$ | $\frac{1-l}{2}+\frac{d}{4}(1+l), \frac{dl}{2}$ | $1-l\left(1-\frac{d}{2}\right), \frac{dl}{2}$ |
|  | ø | $\frac{d}{2}, \frac{dl}{2}$ | $1-l\left(1-\frac{d}{2}\right), \frac{dl}{2}$ | $dW(B_1), dW(B_2)$ |

Figure 3:

While at the present the latter is simply an assumption, later on we will explicitly discuss its validity and we will also to which extent it may be compatible with the PSSPE are characterizing.

The latter assumption on the continuation payoffs implies that, by looking at the payoffs matrix of the response game, there exist two other Nash equilibria of the acceptance game following $p_1^* = p_2^* = \lambda\left(1 - \frac{\delta}{2}\right)$. In both the equilibria, only $B_1$ accepts a price, either $p_1^*$ or $p_2^*$, while $B_2$ always rejects both offers.

Incidentally, note, that if both the above restrictions are violated the acceptance game shows another, different, Nash equilibrium where both buyers reject both offers. Finally, if the first restriction still holds and the second is violated, there are no other Nash equilibria in the response phase than the one where both buyers accept the same price from different sellers.

Therefore, it is clear that if $p_1^* = p_2^* = \lambda\left(1 - \frac{\delta}{2}\right)$ represent equilibrium strategies in the price-offers phase, then the set of strategies we have described is necessarily a PSSPE equilibrium. In fact we have just seen that the strong buyer accepting $p_1^*$, the weak buyer accepting $p_2^*$ definitely constitute a, possibly not unique, Nash equilibrium in the subsequent response game.

Then, we just need to verify that, in the price-offer phase, there are no profitable deviations by any of the sellers from the candidate PSSPE strategies to propose $p_1^* = p_2^* = \lambda\left(1 - \frac{\delta}{2}\right)$.

Consider first seller $S_1$. Given that $S_2$ is offering $p^*$, a possible deviation from her strategy $p_1^* = \lambda\left(1 - \frac{\delta}{2}\right)$ is to propose $p_1' = p_1^* - \varepsilon$. However, it is immediate to see that after such a deviation, the subse-

| | | **B2** | | |
|---|---|---|---|---|
| | | $p_1'$ | $p_2^*$ | ø |
| **B1** | $p_1'$ | $\frac{1-l}{2}+\frac{d}{4}(1+l)-\frac{e}{2},\ \frac{dl}{2}-\frac{e}{2}$ | $1-l\left(1-\frac{d}{2}\right)-e,\ \frac{dl}{2}$ | $1-l\left(1-\frac{d}{2}\right)-e,\ \frac{dl}{2}$ |
| | $p_2^*$ | $1-l\left(1-\frac{d}{2}\right),\ \frac{dl}{2}-e$ | $\frac{1-l}{2}+\frac{d}{4}(1+l),\ \frac{dl}{2}$ | $1-l\left(1-\frac{d}{2}\right),\ \frac{dl}{2}$ |
| | ø | $\frac{d}{2},\ \frac{dl}{2}-e$ | $1-l\left(1-\frac{d}{2}\right),\ \frac{dl}{2}$ | $dW(B_1),\ dW(B_2)$ |

Figure 4:

quent response game would have a single pure Nash equilibrium where $B_1$ accepts $p_2^*$ and the weak buyer accepts $p_1'$. Then the new price offered by $S_1$ would surely be accepted, which, in turn, clearly gives her a lower payoff than the original one. Therefore $p_1' = p_1^* - \varepsilon$ can never represent a profitable deviation.

Consider, at the contrary, that, given that $S_2$ is offering $p^*$, $S_1$ deviates from her strategy by proposing $p_1'' = p_1^* + \varepsilon$, which seems potentially profitable. However, by plugging the new price offer $p_1'' = \lambda\left(1 - \frac{\delta}{2}\right) + \varepsilon$ in the payoff matrix of the generated acceptance game in Figure 4, it is easy to work out the Nash equilibria of the subsequent response phase.

In fact, for $\varepsilon$ not too high, it still holds that $1 - \lambda + \frac{\delta\lambda}{2} - \varepsilon > \frac{1-\lambda}{2} + \frac{\delta}{4}(1+\lambda)$. Therefore, given our assumption on the continuation payoffs, it is easy to see that the response game generated by the deviation of $(p_1'', p_2^*)$ proposals shows two co-existing Nash equilibria.

In one Nash equilibrium, the strong buyer accepts the offer $p_1''$ while the weak buyer accepts $p_2^*$. This Nash equilibrium clearly implies that, by deviating from the candidate equilibrium, $S_1$ can indeed experience a profitable alternative.

However, there exists another Nash equilibrium where the strong buyer accepts the offer $p_2^*$, while the weak buyer rejects both offers. Therefore, in such an equilibrium, the deviating $S_1$ will certainly be matched with delay in a bilateral negotiation with the weak buyer. This, in turn makes her unambigously worse off than adopting the candidate equilibrium strategy, as, in such a case, she can only obtain $\frac{\delta\lambda}{2} < 1 - \lambda + \frac{\delta\lambda}{2}$.

Therefore, what can we conclude about the above deviation $p_1'' = \lambda\left(1 - \frac{\delta}{2}\right) + \varepsilon$ by seller $S_1$? Clearly, the emergence of multeplicity of

Nash equilibria in the generated response phase makes sharp conclusions difficult to be drawn. However, some considerations may be in order.

On the one hand, it may well be argued that one equilibrium Pareto dominates the other. This would push the equilibrium selection problem in favour of only picking the dominant equilibrium, where the strong buyer accepts $p_2^*$, thus making the deviation always unprofitable.

On the other hand, it can also be argued that forcing, by mean of an ad hoc assumption, the weak buyer to always accept an offer whenever indifferent between accepting or rejecting it, it would destroy the equilibrium with delay, thus making the deviation to the surviving equilibrium profitable.

More generally, here the issues of multeplicity of equilibria and equilibrium selection can be seen as even less trivial problems by adding the consideration that, despite being subgame perfect, the above ones are not trembling-hand equilibria.

In order to find a reasonable compromise between all these diverging views, we have attempted to draw the following conclusion.

Let us assume that the response game reaches the equilibrium where the strong buyer accepts $p_2^*$ and the weak buyer rejects both offers with some positive, though possibly infinitesimal, probability $\varphi$, while it reaches the equilibrium where both buyers accept a proposal with the complementary probability.

Then, a sufficient condition for the above described set of strategies to indeed be a PSSPE equilibrium is that $\varphi \geq \varepsilon$, with $\varepsilon$ being the largest of the potentially profitable increase in the price $p^* = \lambda\left(1 - \frac{\delta}{2}\right)$.

In fact, this conjecture is just one, intuitive, condition, from many, to guarantee that the threat to possibly reach the equilibrium with delay in the response game is credible enough to dissuade $S_1$ from attempting to propose a higher price in order to attract the strong buyer.

This conjecture thus rules definitely out the existence of any possible profitable deviation by $S_1$. The same argument as above can of course be extended to show the impossibility of any profitable deviation by $S_2$. In this case, raising the price to $p_2'' = \lambda\left(1 - \frac{\delta}{2}\right) + \varepsilon$ would lead, in the following response game, to either reach an equilibrium where the strong buyer accepts $p_2''$ and the weak accepts $p_1^* = \lambda\left(1 - \frac{\delta}{2}\right)$, or one where the strong buyer accepts the $p_1^*$ while the weak rejects both.

Therefore, we may conclude that, for general values of the parameters $\delta$ and $\lambda$, there are no profitable deviations for any trader. This, in turn proves that the set of strategies according to which both sellers propose an identical price $p^* = \lambda\left(1 - \frac{\delta}{2}\right)$, the strong buyer accepts the offer from $S_1$, and the weak buyer accepts the one from $S_2$, is in fact a PSSPE equilibrium of the $S$-games in the negotiation stage.

## 6.5 Proof of Claim 5

If each seller indeed rejects any price proposed by the buyers, in the following round of bargaining with one-half probability the sellers will be selected to make offers. As we have shown in Claim 4, in any $S$-game of the negotiation stage, there is a PSSPE equilibrium where, in the price-offers phase, both sellers propose the same price $p^* = \lambda\left(1 - \frac{\delta}{2}\right)$, and in the response phase, each buyer accepts it from a different seller. This possibility gives each seller an expected payoff of $\frac{\lambda}{2}\left(1 - \frac{\delta}{2}\right)$, while the buyers $B_1$ and $B_2$ expect payoffs equal to $\frac{\left[1-\lambda\left(1-\frac{\delta}{2}\right)\right]}{2}$ and $\frac{\delta\lambda}{4}$, respectively. On the other hand, given the stationarity of the strategies, in the following round of negotiations there is also one-half probability to enter a $B$-game where both sellers again reject any proposal.

Therefore, each seller expects a continuation payoffs from entering a new round of the negotiation stage equal to $V(S_i) = \frac{\lambda}{2-\delta}\left(1 - \frac{\delta}{2}\right)$, while the weak buyer expects a continuation value of $W(B_2) = \frac{\delta\lambda}{2(2-\delta)}$. But, then, it certainly exists a profitable deviation for the weak buyer. In fact, whenever he has been selected to make an offer, he would be better off by proposing a price $p_2' = \lambda\left(1 - \frac{\delta}{2}\right)$ rather than making unacceptable proposals.

In fact, given that the strong buyer is making an unacceptable offer, by proposing $p_2' = \lambda\left(1 - \frac{\delta}{2}\right)$ the weak buyer would generate a response game where at least one seller will surely accept $p_2'$. In fact, even if both sellers accept the same price $p_2'$, they will expect a payoff of $\frac{\lambda}{2}\left(1 - \frac{\delta}{2} + \frac{\delta}{4}\right)$, which is always greater than $V(S_i) = \frac{\lambda}{2-\delta}\left(1 - \frac{\delta}{2}\right)$. Of course, if only one seller accepts $p_2'$, she will also receive an higher reward than rejecting both offers.

But, knowing that a proposal $p_2'$ will always be accepted at least by one seller, makes such a strategy a profitable deviation for the weak buyer, as $\lambda - p_2' = \frac{\delta\lambda}{2} > W(B_2) = \frac{\delta\lambda}{2(2-\delta)}$.

## 6.6 Proof of Claim 6

Notice that, if this allocation was a PSSPE equilibrium, it would be the case that the following conditions were satisfied.

First, it must be the case that $p_2 < p_1$. In fact if it was that $p_2 > p_1$, seller $S_1$ would have accepted the higher price $p_2$ instead. Again, if the buyers were proposing the same price, so that $p_2 = p_1$, then, given that $S_1$ accepts $p_1$, in an equilibrium the symmetry by the sellers would ensure that $S_2$ will accept $p_2$. Note that, as $p_2 < p_1$, from the same argument of the sellers' symmetry we should expect also $S_2$ will accept $p_2$, which in fact intuitively contradicts the present allocation being an equilibrium.

Second, for the price $p_1$ to be accepted by seller $S_1$ it must be set to a level such that the latter is indifferent between accepting it, gaining $p_1$, and rejecting it going to a further negotiation round in a situation such as the one described in the Fifth Class.

Third, must be the case that, by rejecting both offers, buyer $S_2$ expected an higher payoff than by accepting one of the two. In particular, if seller $S_2$ accepted the same offer $p_1$, he would be randomly selected with probability $\frac{1}{2}$ to sell the good rather than going to bilateral negotiations with the remaining buyer. Then if this was an equilibrium it must be that the expected payoff for $S_2$ by accepting $p_1$ would never be as high as the payoff he may obtain by rejecting and going directly to bilateral negotiation with $B_2$, that is the following must hold: $\frac{1}{2}p_1 + \frac{1}{2}\left(\delta\frac{\lambda}{2}\right) \le \delta\frac{\lambda}{2}$.

The latter implies that the emerging price would be such that $p_1 \le \delta\frac{\lambda}{2}$, which in turn also implies, with the first condition, that $p_2 < \delta\frac{\lambda}{2}$, a condition that guarantees that buyer $S_2$ never accepted price $p_2$.

Then it may be shown that the present allocation can never constitute a PSSPE equilibrium. (Rewrite the Proof from here on).

In fact, consider buyer $B_2$. If the present allocation was the equilibrium he would earn a surplus $\delta\frac{\lambda}{2}$. Consider now a deviation by $B_2$ from the described strategy. For instance, he may deviate by proposing a price $p_2' = p_1 + \varepsilon$, with $\varepsilon$ infinitesimally small and, clearly, $p_1 \le \delta\frac{\lambda}{2}$. We now show that this is indeed a profitable deviation, as $B_2$ will surely convince $S_1$ to sell him the good at that price, being able to earn $\lambda - p_1 - \varepsilon$ rather than the lower $\delta\frac{\lambda}{2}$. In fact, the condition $p_1 \le \delta\frac{\lambda}{2}$ implies that $p_2' = p_1 + \varepsilon \le \delta\frac{\lambda}{2} + \varepsilon$ and then that $\lambda - p_1 - \varepsilon \le \lambda - \delta\frac{\lambda}{2} - \varepsilon$. However, as $\varepsilon$ is so small that $\varepsilon \le \lambda(1 - \delta)$, that is $\varepsilon \to 0$ as $\delta \to 1$, it always holds that $\lambda - p_1 - \varepsilon \ge \delta\frac{\lambda}{2}$, which in turn implies that, by proposing $p_2'$, $B_2$ may convince seller $S_1$ to sell him the good and can profitably deviate from the original situation.

Thus, the described allocations can never constitute a subgame perfect equilibrium, because of two forces. On the one hand, the symmetry among the sellers makes impossible for the buyers to offer a price that would be accepted by only one of them; on the other hand, both buyers have incentives to propose offers that guarantee themselves a payoff no lower than in bilateral negotiations.

For the other three cases in the figure, the same logic applies, with the important consideration that, such as in an auction, the high-valuation buyer may always offer a higher price than the weak buyer.

## 6.7 Proof of Claim 7

The proof is immediate. In fact, if the present was an equilibrium, the condition $\frac{1}{2}p_2 + \frac{1}{2}\left(\frac{\delta}{2}\right) \ge \frac{\delta}{2}$ would necessarily hold since each seller would be better off by accepting the price $p_2$ , given that also the other seller is accepting it, rather than rejecting both the offers, then going to bilateral negotiation with the high-valuation buyer. The latter condition would imply $p_2 \geq \frac{\delta}{2} > \delta\frac{\lambda}{2}$.

Furthermore, note that if this was an equilibrium, it would also necessarily be that $p_1 < p_2$. In fact, if, at the contrary, it was that $p_1 > p_2$, then both sellers would have accepted $p_1$ instead. If, again, it was that $p_1 = p_2$, one of the two seller, given the choice of the other, would have preferred to deviate by accepting $p_1$, rather than $p_2$: in fact, in this case, the condition $p_2 \geq \frac{\delta}{2}$ immediately implies that $p_1 \geq \frac{1}{2}p_2 + \frac{1}{2}\left(\frac{\delta}{2}\right)$.

Again, notice that if the present was an equilibrium, buyer $B_2$ would have chosen a price $p_2 = \frac{\delta}{2}$, as any higher price, while still accepted by both sellers, would imply a lower surplus for himself.

Hence, it may be observed that the present allocation can never be an equilibrium. In fact, the high-valuation buyer may always profitably deviate by proposing a price $p_1' = p_2 + \varepsilon$, with $\varepsilon > 0$: in fact, by doing so he may obtain a payoff of $1 - \frac{\delta}{2} - \varepsilon$ which is greater than the one he would get in bilateral negotiation as $1 - \varepsilon > \delta$ for $\varepsilon$ small enough.

## 6.8 Proof of Proposition 8

We treat first the case $\lambda < \frac{\delta}{4-3\delta}$.

In the present candidate PSSPE equilibrium, the sellers are both accepting the proposal from the strong buyer and, are then expecting to be matched with one-half probability with the weak buyer in the next round. By doing so, thay both earn an expected payoff equal to $p_2^* = \lambda\left(1 - \frac{\delta}{2}\right)$. As the equilibrium price proposed by the strong buyer in the price-offer phase can be rewritten as $p_1^* = \frac{\lambda}{2}(4 - 3\delta)$, buyers $B_1$ and $B_2$, thus, expect a surplus of $1 - \frac{\lambda}{2}(4 - 3\delta)$ and $\frac{\delta\lambda}{2}$, respectively.

Then, we first need to check that the Nash equilibrium in the response phase following the above two proposals $p_1^*$ and $p_2^*$ is such that both sellers accept the offer from the strong buyer.

In the following, we will make an assumption over the continuation payoffs expected by the sellers in case, in the response game, they all reject both the proposals, thus entering a new negotiation stage. In fact, we assume that the expected continuation payoff $\delta V(S_i)$ for seller $i = 1, 2$ is such that $\delta V(S_i) < \min\left\{\frac{\delta}{2}, \frac{\lambda}{2}(4 - 3\delta)\right\}$. While at the present the latter is simply an assumption, later on we will explicitly discuss its validity and we will also to which extent it may be compatible with the PSSPE are characterizing.

Hence, we can represent the response game generated by the proposals $p_1^*$ and $p_2^*$ by means of the normal form in Figure 5:

Therefore, it may seen that, whenever any of the sellers accepts the proposal from the strong buyer, the remaining seller is indifferent be-

|  |  | $S_2$ |  |  |
|---|---|---|---|---|
|  |  | $p_1$ | $p_2$ | ø |
| $S_1$ | $p_1$ | $l\left(1-\frac{d}{2}\right), l\left(1-\frac{d}{2}\right)$ | $\frac{l}{2}(4-3d), l\left(1-\frac{d}{2}\right)$ | $\frac{l}{2}(4-3d), \frac{dl}{2}$ |
|  | $p_2$ | $l\left(1-\frac{d}{2}\right), \frac{l}{2}(4-3d)$ | $\frac{1}{2}\left[l\left(1-\frac{d}{2}\right)+\frac{d}{2}\right], \frac{1}{2}\left[l\left(1-\frac{d}{2}\right)+\frac{d}{2}\right]$ | $l\left(1-\frac{d}{2}\right), \frac{d}{2}$ |
|  | ø | $\frac{dl}{2}, \frac{l}{2}(4-3d)$ | $\frac{d}{2}, l\left(1-\frac{d}{2}\right)$ | $dV(S_1), dV(S_2)$ |

Figure 5:

tween either to accept $p_1^*$ as well, or to accept the offer from the weak buyer. Both sellers accepting $p_1^*$ is in fact a Nash equilibrium of the response game.

Moreover, it can be checked that there are no other Nash equilibria in such a game. In particular, given that one seller accepts $p_2^*$, it is always better for the other seller to reject both offers and to wait the next round to start bilateral negotiations with the strong buyer. In fact, as long as $\lambda < \frac{\frac{\delta}{2}}{1-\frac{\delta}{2}}$, it always holds that $\frac{\delta}{2} > \frac{\lambda\left(1-\frac{\delta}{2}\right)}{2} + \frac{\delta}{4}$. Moreover, as long as $\lambda < \frac{\delta}{4-3\delta}$, it is also verified that $\frac{\delta}{2} > \frac{\lambda}{2}(4-3\delta)$.

The two above conditions may be summarized in the following $\lambda < \min\left\{\frac{\frac{\delta}{2}}{1-\frac{\delta}{2}}, \frac{\delta}{4-3\delta}\right\}$, which is trivially verified as $\delta \to 1$, and it is always holding as the traders are sufficiently patient, that is as $\delta > \bar{\delta}$. Furthermore, it is also immediately checked that $\frac{\frac{\delta}{2}}{1-\frac{\delta}{2}} > \frac{\delta}{4-3\delta}$ is always holding. Therefore, the latter just reduces to $\lambda < \frac{\delta}{4-3\delta}$, the condition we set in the first part of our Proposition. Also note that condition $\lambda < \frac{\delta}{4-3\delta}$ implies that our previous assumption on the continuation payoffs of the sellers reduces to $\delta V(S_i) < \frac{\lambda}{2}(4-3\delta)$.

Finally, given that one seller decides to reject both offers, our assumption on the continuation payoffs guarantees that the best response by the other seller will always be to accept the proposal from the strong buyer.

This, in turn, concludes the proof that are no other Nash equilibria in the response game but the one where both sellers accept $p_1^* = \frac{\lambda}{2}(4-3\delta)$

from the strong buyer.

As a second step, we need to verify that, in the price-offer phase, there are no profitable deviations by the strong and the weak buyer from the candidate PSSPE strategies to propose $p_1^* = \frac{\lambda}{2}(4-3\delta)$ and $p_2^* = \lambda\left(1-\frac{\delta}{2}\right)$ respectively.

Consider first the low-valuation buyer. Immagine $B_2$ deviates by proposing a higher price $p_2' = p_2^* + \varepsilon$. Given that one seller is accepting $p_1^*$, a proposal $p_2'$ is such that it ensures to the seller accepting it a payoff $\lambda\left(1-\frac{\delta}{2}\right) + \varepsilon$, clearly higher than what gained when he is also choosing $p_1^*$.

Therefore, in the following response phase, there will be a mixed strategies equilibrium where with some non-null probability $\pi$ at least one seller would accept the offer $p_2'$. However, such a deviation is clearly not profitable for $B_2$ as he gets $(1-\pi)\frac{\delta\lambda}{2} + \pi\left[\lambda - \lambda\left(1-\frac{\delta}{2}\right) - \varepsilon\right] = \frac{\delta\lambda}{2} - \pi\varepsilon$, which is obviously lower than the surplus he may gain in bilateral negotiation in the original PSSPE candidate.

Immagine, on the other hand, that $B_2$ deviates by proposing a lower price $p_2' = p_2^* - \varepsilon$. It is easy to see that, in the following response phase, there still is a unique Nash equilibrium where both the sellers accept the offer from the strong buyer. As the weak buyer will then expect the same payoff $\frac{\delta\lambda}{2}$ from the subsequent bilateral negotiation, also such a deviation is clearly not profitable. Hence, $B_2$ indeed does not have any profitable deviation from the above candidate PSSPE equilibrium.

Consider finally the high-valuation buyer. If $B_1$ deviates and makes an higher price $p_1' = p_1^* + \varepsilon$, the offer will be immediately accepted by both sellers, but $B_1$ will end up paying an higher price, which is clearly not profitable.

On the other hand, what happens if $B_1$ deviates and offers a lower price $p_1' = p_1^* - \varepsilon$? By looking at the generated response game as depicted in the normal form in Figure 6, it is immediate to see that there are no pure strategies Nash equilibria in the response phase.

We are then looking for symmetric mixed strategies Nash equilibrium. Let each seller play "Accept $p_1'$" with probability $\rho$, "Accept $p_2^*$" with probability $\sigma$, and "Reject both offers" with the residual probability $1-\rho-\sigma$. Hence, by letting each seller randomize over her strategies in a way to equalize her expected payoff, we obtain the equalities

$$\frac{\lambda}{2}(4-3\delta) - \varepsilon + \rho\left[\frac{\varepsilon}{2} + \lambda(\delta-1)\right] = \lambda\left(1-\frac{\delta}{2}\right) + \frac{\sigma}{2}\left[\frac{\delta}{2} - \lambda\left(1-\frac{\delta}{2}\right)\right] =$$

$$= \rho\frac{\delta\lambda}{2} + \sigma\frac{\delta}{2} + (1-\rho-\sigma)\delta V(S_i)$$

Solving the latter gives the cumbersome expressions for the non-degenerate probabilities $\rho^*$ and $\sigma^*$. Note that it is only when $\delta \to 1$ that $\rho^* = 1$

|  |  | **S₂** |  |  |
|---|---|---|---|---|
|  |  | $p_1$ | $p_2$ | $\varnothing$ |
| **S₁** | $p_1$ | $l\left(1-\frac{d}{2}\right)-\frac{e}{2},\ l\left(1-\frac{d}{2}\right)-\frac{e}{2}$ | $\frac{l}{2}(4-3d)-e,\ l\left(1-\frac{d}{2}\right)$ | $\frac{l}{2}(4-3d)-e,\ \frac{dl}{2}$ |
|  | $p_2$ | $l\left(1-\frac{d}{2}\right),\ \frac{l}{2}(4-3d)-e$ | $\frac{1}{2}\left[l\left(1-\frac{d}{2}\right)+\frac{d}{2}\right],\ \frac{1}{2}\left[l\left(1-\frac{d}{2}\right)+\frac{d}{2}\right]$ | $l\left(1-\frac{d}{2}\right),\ \frac{d}{2}$ |
|  | $\varnothing$ | $\frac{dl}{2},\ \frac{l}{2}(4-3d)-e$ | $\frac{d}{2},\ l\left(1-\frac{d}{2}\right)$ | $dV(S_1), dV(S_2)$ |

Figure 6:

and $\sigma^* = 0$: that is, both sellers always accept the price offered from the strong buyer just in the frictionless case. in all the other cases randomizing with positive probabilities over all the strategy set.

Therefore, we can easily work out the strong buyer's expected payoff from the response game following his deviation $p_1^0 = p_1^* - \varepsilon$. Denote $\delta W(B_1) < 1$ his continuation payoff from entering a new round of the negotiation stage after the two sellers have rejected both the proposals.

Then, the expected payoff of $B_1$ from the response game following his deviation $p_1^0 = p_1^* - \varepsilon$ is higher than the surplus expected from the above candidate PSSPE only if

$$\left[\rho^{*2} + 2\rho^*\sigma^* + 2\rho^*(1-\rho^*-\sigma^*)\right]\left(1 - \tfrac{\lambda}{2}(4-3\delta) + \varepsilon\right) + \\ + \left[\sigma^{*2} + 2\sigma^*(1-\rho^*-\sigma^*)\right]\tfrac{\delta}{2} + (1-\rho^*-\sigma^*)^2\,\delta W(B_1) < 1 - \tfrac{\lambda}{2}(4-3\delta)$$

However, by substituting the expressions for $\rho^*$ and $\sigma^*$, it can be checked that the latter inequality is always violated unless $\delta \to 1$ and $\rho^* \to 1$.

Hence, having shown that there are no profitable deviations for the strong buyer as well, concludes the proof that the above described set of strategies by the traders is indeed a PSSPE of the negotiation stage for $\lambda < \frac{\delta}{4-3\delta}$.

We now address our attention to the case where $\lambda > \frac{\delta}{4-3\delta}$.

In this candidate PSSPE equilibrium, both sellers are accepting the proposal from the strong buyer and, are then expecting to be matched with one-half probability with the weak buyer in the next round. By doing so, they both expect a payoff equal to $p_2^* = \frac{\delta}{4}(1+\lambda)$. Buyers $B_1$ and $B_2$ expect a surplus of $1 - \frac{\delta}{2}$ and $\frac{\delta\lambda}{2}$, respectively.

| | | $S_2$ | | |
|---|---|---|---|---|
| | | $p_1^*$ | $p_2^*$ | ø |
| $S_1$ | $p_1^*$ | $\frac{d}{4}(1+l), \frac{d}{4}(1+l)$ | $\frac{d}{2}, \frac{d}{4}(1+l)$ | $\frac{d}{2}, \frac{dl}{2}$ |
| | $p_2^*$ | $\frac{d}{4}(1+l), \frac{d}{2}$ | $\frac{d}{8}(3+l), \frac{d}{8}(3+l)$ | $\frac{d}{4}(1+l), \frac{d}{2}$ |
| | ø | $\frac{dl}{2}, \frac{d}{2}$ | $\frac{d}{2}, \frac{d}{4}(1+l)$ | $V(S_1), V(S_2)$ |

Figure 7:

Again, we first need to check that the Nash equilibrium in the response phase following the above two proposals $p_1^*$ and $p_2^*$ is such that both sellers accept the offer from the strong buyer.

In the following, we will make an assumption over the continuation payoffs the sellers expect from a new round of the negotiation stage after that, in the response game, they all reject both the proposals. Here, we assume that the expected continuation payoff $\delta V(S_i)$ for seller $i = 1, 2$ is such that $\delta V(S_i) < \min\left\{\frac{\delta}{2}, \frac{\delta}{4}(1+\lambda)\right\} = \frac{\delta}{4}(1+\lambda)$. Once again, while at the present the latter is simply an assumption, later on we will explicitly discuss its validity and we will also to which extent it may be compatible with the PSSPE we are characterizing.

As usual, we can represent the response game generated by the proposals $p_1^*$ and $p_2^*$ by means of the normal form in Figure 7:

It may be seen that, provided that $\lambda > \frac{\delta}{4-3\delta}$, whenever any of the sellers accepts the proposal from the strong buyer, the remaining seller is indifferent between either to accept $p_1^*$ as well, or to accept the offer from the weak buyer. Both sellers accepting $p_1^*$ is in fact a Nash equilibrium of the response game.

Moreover, $\frac{\delta}{2} > \frac{3\delta}{8} + \frac{\delta\lambda}{8}$ implies other two asymmetric Nash equilibria there exist in the response game following the proposals $p_1^*$ and $p_2^*$. In both, one of the sellers accepts the offer from the strong buyer, while the other accepts the price proposed by the weak buyer. These in fact are the two asymmetric PSSPE equilibria with no delay in the trade we will characterize in the next Proposition. Note that also in these equilibria, buyers $B_1$ and $B_2$ expect a surplus of $1 - \frac{\delta}{2}$ and $\frac{\delta\lambda}{2}$, respectively.

We then need to verify that, in the price-offer phase, there are no prof-

| | | $S_2$ | | |
|---|---|---|---|---|
| | | $p_1^*$ | $p_2^{\prime}$ | ø |
| $S_1$ | $p_1^*$ | $\frac{d}{4}(1+l), \frac{d}{4}(1+l)$ | $\frac{d}{2}, \frac{d}{4}(1+l)-e$ | $\frac{d}{2}, \frac{dl}{2}$ |
| | $p_2^{\prime}$ | $\frac{d}{4}(1+l)-e, \frac{d}{2}$ | $\frac{d}{8}(3+l)-\frac{e}{2}, \frac{d}{8}(3+l)-\frac{e}{2}$ | $\frac{d}{4}(1+l)-e, \frac{d}{2}$ |
| | ø | $\frac{dl}{2}, \frac{d}{2}$ | $\frac{d}{2}, \frac{d}{4}(1+l)-e$ | $V(S_1), V(S_2)$ |

Figure 8:

itable deviations by the strong and the weak buyer from the candidate PSSPE strategies to propose $p_1^* = \frac{\delta}{2}$ and $p_2^* = \frac{\delta}{4}(1+\lambda)$ respectively.

Clearly any alternative proposal higher than $p_1^*$ and $p_2^*$ will never be profitable for the strong and the weak buyer, respectively, since, being immediately accepted at least by one seller, it will imply lower surplus.

Consider first the weak buyer. Immagine $B_2$ deviates by proposing a lower price $p_2^{\prime} = p_2^* - \varepsilon$. It is easy to see in Figure 8 that, in the following response phase, there is a unique Nash equilibrium where both the sellers accept the offer from the strong buyer:

As the weak buyer will then expect the same payoff $\frac{\delta\lambda}{2}$ from the subsequent bilateral negotiation, such a deviation is clearly not profitable. Hence, $B_2$ indeed does not have any profitable deviation from the above candidate PSSPE equilibrium.

Consider then the high-valuation buyer. Immagine $B_1$ deviates and offers a lower price $p_1^{\prime} = p_1^* - \varepsilon$. By looking at the generated response game as depicted in the normal form in Figure 9, it is easy to see that there are no pure strategies Nash equilibria in the response phase.

Again, we should then look for symmetric mixed strategies Nash equilibrium. Let each seller play "Accept $p_1^{\prime}$" with probability $\phi$, "Accept $p_2^*$" with probability $\psi$, and "Reject both offers" with the residual probability $1 - \phi - \psi$. Hence, by letting each seller randomize over her strategies in a way to equalize her expected payoff, we obtain the equalities

$$\phi\left(\frac{\delta}{4}(1+\lambda) - \frac{\varepsilon}{2}\right) + (1-\phi)\left(\frac{\delta}{2} - \varepsilon\right) = \psi\left(\frac{3\delta}{8} + \frac{\delta\lambda}{8}\right) + (1-\psi)\frac{\delta}{4}(1+\lambda) =$$

$$= \varphi\frac{\delta\lambda}{2} + \psi\frac{\delta}{2} + (1-\varphi-\psi)\,\delta V(S_i)$$

| | | $S_2$ | | |
|---|---|---|---|---|
| | | $p_1'$ | $p_2^*$ | Ø |
| $S_1$ | $p_1'$ | $\frac{d}{4}(1+l)-\frac{e}{2}, \frac{d}{4}(1+l)-\frac{e}{2}$ | $\frac{d}{2}-e, \frac{d}{4}(1+l)$ | $\frac{d}{2}-e, \frac{dl}{2}$ |
| | $p_2^*$ | $\frac{d}{4}(1+l), \frac{d}{2}-e$ | $\frac{d}{8}(3+l)-\frac{e}{2}, \frac{d}{8}(3+l)-\frac{e}{2}$ | $\frac{d}{4}(1+l)-e, \frac{d}{2}$ |
| | Ø | $\frac{dl}{2}, \frac{d}{2}-e$ | $\frac{d}{2}, \frac{d}{4}(1+l)-e$ | $dV(S_1), dV(S_2)$ |

Figure 9:

Solving the latter gives some cumbersome expressions for the non-degenerate probabilities $\phi^*$ and $\psi^*$.

Therefore, we can easily work out the strong buyer's expected payoff from the response game following his deviation $p_1' = p_1^* - \varepsilon$. Denote $\delta W(B_1) < 1$ his continuation payoff from entering a new round of the negotiation stage after the two sellers have rejected both the proposals.

Then, the expected payoff of $B_1$ from the response game following his deviation $p_1' = p_1^* - \varepsilon$ is higher than the surplus expected from the above candidate PSSPE only if

$$\left[\phi^{*2} + 2\phi^*\psi^* + 2\phi^*(1-\phi^*-\psi^*)\right]\left(1-\frac{\delta}{2}+\varepsilon\right) + \left[\psi^{*2} + 2\psi^*(1-\phi^*-\psi^*)\right]\frac{\delta}{2} + (1-\phi^*-\psi^*)^2\,\delta W(B_1) < 1-\frac{\delta}{2}.$$

However, by substituting the expressions for $\phi^*$ and $\psi^*$, it can be checked that the latter inequality is always violated unless $\delta \to 1$.

Hence, having shown that there are no profitable deviations for the strong buyer as well, concludes the proof that the above described set of strategies by the traders is indeed a PSSPE of the negotiation stage for $\lambda > \frac{\delta}{4-3\delta}$.

## 6.9 Proof of Proposition 9

To see that the above strategies are indeed a PSSPE for $\lambda > \frac{\delta}{4-3\delta}$, consider first the sellers. In the above candidate equilibrium allocation, $S_1$ would get a profit of $\frac{\delta}{2}$, while $S_2$ would gain a payoff of $\frac{\delta}{4}(1+\lambda)$.

Consider the response game generated by the offers $p_1^* = \frac{\delta}{2}$ and $p_2^* = \frac{\delta}{4}(1+\lambda)$; this is the same described in the Proof of Proposition 8, and represented in TABLE 5

It is immediate to check that, given the price offers by the buyers and the choice of the other seller, each seller can not experience any profitable deviation.

In fact, given that $S_2$ is accepting $p_2^*$, $S_1$ could possibly deviate either by rejecting both offers, or by also accepting $p_2^*$ instead. However, in the former case, as she would enter bilateral negotiations with the strong buyer, she would expect exactly the same payoff of $\frac{\delta}{2}$. In the latter case, she would compete with $S_2$ in order to trade with the weaker buyer, being half of the times matched with $B_1$ himself to start a bilateral negotiation with one period of delay: thus, clearly $\frac{\delta}{2} > \frac{1}{2}\left[\frac{\delta}{2} + \frac{\delta}{4}(1+\lambda)\right]$.

Analogously, $S_2$ can never profitably deviate by accepting $p_2^*$, given that $S_1$ is accepting $p_1^*$. In fact, if she would reject both offers, she would start a bilateral negotiation with the weak buyer himself, thus getting an expected payoff of $\frac{\delta\lambda}{2}$ which is strictly lower than $\frac{\delta}{4}(1+\lambda)$ since $\lambda > \frac{\delta}{4-3\delta}$. On the other hand, if she would also accept $p_1^*$, from the subsequent random draw, she would earn exactly the same profit $\frac{1}{2}\left[\frac{\delta}{2} + \frac{\delta\lambda}{2}\right]$.

As already shown in the Proof of Proposition 8, $S_1$ accepting $p_1^*$, $S_2$ accepting $p_2^*$ is in fact one of the two asymmetric Nash equilibria coexisting in the response game with the symmetric equilibrium in which both sellers accept the offer from the strong buyer.

Consider, now the buyers in the price-offerg phase. In the candidate equilibrium, the strong buyer would earn an expected surplus of $1 - \frac{\delta}{2}$ while the weak buyer would expect a surplus of $\lambda - \frac{\delta}{4}(1+\lambda)$.

Consider now the possible deviations available to the buyers. Clearly any alternative proposal higher than $p_1^*$ and $p_2^*$ will never be profitable for the strong and the weak buyer, respectively, since, being immediately accepted at least by one seller, it will imply lower surplus.

Consider first the weak buyer. Immagine $B_2$ deviates by proposing a lower price $p_2' = p_2^* - \varepsilon$. As already shown in Figure 8 in the Proof of Proposition 8, in the following response phase, there is a unique Nash equilibrium where both the sellers accept the offer from the strong buyer. In such a case the weak buyer will then expect the same payoff $\frac{\delta\lambda}{2}$ from the subsequent bilateral negotiation.

However, as $\lambda > \frac{\delta}{4-3\delta}$, it always holds that $\lambda - \frac{\delta}{4}(1+\lambda) > \frac{\delta\lambda}{2}$, and that, therefore, such a deviation would not be profitable. Hence, $B_2$ indeed does not have any profitable deviation from the above candidate PSSPE equilibrium.

Consider then the high-valuation buyer. Immagine $B_1$ deviates and offers a lower price $p_1' = p_1^* - \varepsilon$. By looking at the generated response game as depicted in the normal form in Figure 9 in the Proof of Proposition 8, it is easy to see that there are no pure strategies Nash equilibria in the response phase.

Hence, the same line of arguments developed in the Proof of Proposition 8 can be repeated to show that neither such a deviation would ever be pro…table for the strong buyer.

Therefore this concludes the proof that the above set of strategies in the price-o¤ers and response phases indeed represent a PSSPE of the negotiation stage for $\lambda > \frac{\delta}{4 - 3\delta}$.

## 7 Appendix. Acknowledgements

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