Learning with Expert Advice*

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September 3, 2004

Abstract

Surveys on inflation forecasts show that expectations combine forward-looking and backward-looking elements. In our paper we rationalize this finding in an equilibrium framework. We assume two types of agents, one having rational expectations and the other using adaptive learning in his forecasts, ratio of the two types in the population evolve according to their past forecasting performance. Our first result is that even a misspecified learning algorithm survives competition with rational expectations. Further, the presence of rational agents speeds up convergence of the learning algorithm. These findings may strengthen the case for using learning models enriched with rational agents to model expectations.

Introduction

The importance of forward-looking behavior in economic decision making have long been recognized in economics. However the modelling of expectations remains a matter of controversy. Since Muth 1960 [24] rational expectations have been a received benchmark. Simply stated, rational expectations posits that agents do not make systematic forecast errors. However rational expectations is criticized for placing unreasonable computational and informational demands on economic agents. Also, a vast empirical literature on testing survey data rejects rational expectations and economic models with

*Acknowledgements: I am grateful to Albert Marcet. Special thanks to Ramon Marimon and Massimo Guidolin.

rational expectations often perform very badly on data. These criticisms raised the importance of seeking alternative ways to model expectations.

However, once we depart from fully rational expectations there are many ways to do so. To maintain some consistency the literature on adaptive learning and boundedly rational modelling[1] still attributes a lot of rationality to agents, they "behave as working economists or econometricians"[2]. Still, the choice of a learning algorithm is necessarily arbitrary, and subject to the caveat that agents would eventually abandon their ad-hoc learning rule, if they can do better. Even though under certain conditions adaptive learning can converge to rational expectations, still slow convergence can significantly affect finite sample behavior of the economy. In this sense slow convergence might even cast doubt on the validity of the final limit.

In our paper we address the criticisms of adaptive learning raised above: would learning survive in the presence of a better forecasting algorithm, and how fast learning converges. We raise the question what happens if agents follow least squares learning but they have access to the forecast of an 'expert' who can actually do better. This expert will have rational expectations. We establish a sort of 'forecasting competition' between the least squares learner and the rational agent in a self referential model, and examine whether the learner survives[3]. We are also interested in whether the presence of a well informed agent can "help" to increase rate of convergence.

One main result of our paper is that adaptive learning survives. This means that least squares forecasts are closer to the actual outcome than the forecasts of the rational expert with a positive probability even in the limit. Interestingly, this conclusion holds even if the learning algorithm is misspecified. In other words the weights on least squares learning will not collapse to zero, in equilibrium both learners and rational agents are present in the population. This result rationalizes empirical work on survey expectations which

[1]Marimon 1996 [22] provides a survey of this research.

[2]Sargent 1993 [29] p.22.

[3]The idea of forecasting with the use of expert advice is an idea applied in several branches of economic theory, for a summary see Cesa-Bianchi and Lugosi 2003 [9]. In game theory, the concept of correlated equilibria takes experts to be pure strategies; in finance, portfolio choice models regard experts as different portfolio strategies. In the terminology of this literature experts in our paper would be the two forecasting algorithms (forecasting with least squares regression and rational expectations). To avoid confusion we will call only the rational agent to be an expert. (Experts of this paper are "*simulatable experts*": functions that use data accessible to the forecaster himself, thus the forecaster can simulate the experts' future reactions.)

suggest that in reality expectations combine backward- and forward-looking elements.

A second main result is that the presence of a rational agent increases rate of convergence of least squares learning. Thus, when agents have access to rational forecasts the 'slow convergence criticism' of adaptive learning will not hold.

The novelty of our approach is that the ratio of learners and rational agents in the population is not imposed exogenously, but depend on their past performance. Who made better forecasts in the past will have higher weight. An interesting feature of this weighting is that heterogeneity can be an equilibrium outcome. Literature on heterogeneous learning usually assumes an ad-hoc degree of heterogeneity. See for example Giannitsarou 2001 [16] who studies stability of rational expectations equilibria under different types of heterogeneous learning. Evans, Honkapohja and Marimon 2001 [15] uses stochastic heterogeneity, however here heterogeneity is only transitory, it vanishes in the limit. An exception is Evans, Branch 2003 [6]. They built a heterogeneous learning model with endogenous weights where all learning algorithms are misspecified and the weights on them depend on their unconditional mean payoff. They find heterogeneity in equilibrium, and introduce the concept of Misspecification Equilibrium.

We close the introduction by discussing some related empirical and theoretical research. One strand of empirical research examines survey measures of inflation expectations and finds an intermediate degree or rationality. A vast literature on testing survey data rejects the rational expectations hypothesis. Lovell 1986 [18] reviews evidence against rational expectations from a number of empirical studies. Baghestani 1992 [2] Ball and Croushore 1995 [3] show that survey forecasts do not make efficient use of all available information.

Survey expectations are not purely adaptive neither: they reflect more information than what is embedded in lagged inflation. Mullineaux 1980 [23] for example found that the money supply helps explain movements in survey expectaions, even controlling for lagged inflation. Baghestani 1992 [2] shows that survey data are biased, but contain more predictive information than what is included in the naive forecasts.

Recent surveys detect an intermediate degree of rationality: inflation expectations are well represented as being a weighted average of forward-looking and backward-looking expectations. Roberts 1998 [27] finds that survey measures of inflation expectations are neither perfectly rational nor as unsophisticated as simple autoregressive models. Baak 1999 [1] and Chavas (2000) [10] present empirical evidence for the presence of heterogeneous price expectations (rational and boundedly rational expectations) in the U.S. beef market.

Moreover, modelling expectations as a weighted average of adaptive and rational expectations is supported by an other strand of empirical literature. These papers show that the empirical performance of standard models improves when instead of rational expectations the modelling choice for expectations is a mixture of backward-looking and forward-looking expectations (or survey expectations). Roberts 2001 [28] shows that the New Keynesian model does not fit the U.S. data well unless additional lags of inflation are used, which is not implied by rational expectations. He gives the interpretation to these additional lags that some fraction of the population uses a simple univariate rule for forecasting inflation. Roberts 1997 [26] shows that if surveys are assumed to capture inflation expectations accurately, then there is no need for additional lags of inflation in the New Keynesian Phillips curve. This result implies, that if surveys accurately reflect inflation expectations, it is imperfectly rational expectations and not the underlying structure of the economy that accounts for the presence of lagged inflation in empirical estimates of the New Keynesian model[4].

The modelling choice of expectations is also important for policy evaluation. We already know that using adaptive learning instead of rational expectations has important implications for macroeconomic policy. For example Evans and Honkapohja 2001 [5] [12], Evans and Honkapohja 2002 [14], Bullard and Mitra 2003[6] [7] deal with differences of optimal monetary policy

[4]Survey evidence is subject to the caveat that survey respondents may not have incentives to provide accurate information. So survey expectations are at best a noisy measure of inflation expectations and at worst tell us nothing about actual inflation expectations. However it may boost confidence in the usefulness of survey expectations that they are helpful in modeling inflation and in predicting wages and interest rates (Roberts 1995, 1997 [25], [26], Englander and Stone 1989 [11]). Roberts 1995, 1997 found that using survey expectations instead of rational expectations improves the ability of New Keynesian model to fit the data.

[5]Evans and Honkapohja 2001 [12]finds that a fundamentals based monetary policy rule, which would be the optimal monetary policy without commitment when private agents have perfectly rational expectations, is unstable if agents follow standard adaptive learning rule.

[6]Bullard and Mitra [7] 2000 emphasizes the importance of the Taylor principle in obtaining stable and determinate interest rate rules.

under learning and under rational expectations. Evans and Honkapohja 2002 [13] examines interactions between fiscal and monetary policy under learning and under rational expectations. Marcet and Nicolini 2003 [21] shows that a standard monetary model with least squares learning can reproduce stylized facts of recurrent hyperinflations in the 80's . Our paper would like to support modelling expectations as a mixture of backward- and forward looking expectations. We find it important for future research to examine policy implications of this modelling choice.

Finally our paper shows that the rate of convergence of least squares learning is faster in the presence of a rational agent. This is important since in practice slow convergence can even mean no convergence ar all, for example when there are regime shifts or changes in policy and it takes several years for agents to learn the new equilibrium. Research on speed of convergence is relatively scarce even though slow convergence effects real economic activity and also has policy implications. In the first original contribution Marcet and Sargent 1995 [20] analyzes convergence of least squares learning. They find that convergence to rational expectations can be so slow that the standard asymptotic distributions used in classical econometrics will not obtain. For Bayesian learning Vives 1993 [31] finds slow rate of convergence. Similarly to our results he shows that in the presence a positive mass of perfectly informed agents Bayesian learning achieves $\sqrt{t}$ convergence. We believe, our speed of convergence results support modelling adaptive learning with rational expectations since the slow convergence problem can be avoided.

Section 1 presents the model. Section 2 establishes properties of the equilibrium and conditions of convergence to the equilibrium. Section3 examines speed of convergence. We compare speed of convergence results to least squares learning and derive the conditions under which the presence of a well informed expert can speed up convergence to equilibrium in our framework. Then, section 4 provides some numerical results on finite sample speed of convergence with Monte Carlo simulations.

1 The Model

The starting point of the analysis is a simple self-referential model. The name self-referential, comes from the fact that there is a feedback from expectations to actual outcomes of the endogenous variable. Keeping the model simple this paper focuses on the expectations side.

Let the endogenous variable p to be the price level, and suppose it is determined by the price level expected for the next period and on the nominal money stock m. Here the money stock is exogeneous and follows an AR(1) process.

$$p_t = \lambda \widetilde{E}_t p_{t+1} + m_t \quad (1)$$
$$m_t = \varrho m_{t-1} + \varepsilon_t \qquad \lambda \neq 0,\ \varrho \in [0,1),\ \varepsilon \sim IID(0, \sigma_\varepsilon^2) \quad (2)$$

Equation (1) and (2) together with the assumptions about the expectation term completely determine the price level. This corresponds to a simple version of the Cagan model of inflation (1956) [8][7].

We will suppose there are two types of agents, one following least squares learning (henceforth LS) and one having rational expectation (henceforth RE). Let us denote the expectation of least squares learners by E^{LS} and the expectation of the rational agents by E^{RE}. Then the aggregate expectation about the price level of next period $\widetilde{E}_t p_{t+1}$ is formed as an average of the forecasts of the two types of agents:

$$\widetilde{E}_t p_{t+1} = \omega_t E_t^{LS} p_{t+1} + (1-\omega_t) E_t^{RE} p_{t+1} \qquad \omega_t \in [0,1] \quad (3)$$

Weights in the aggregate expectation -ω_t- will evolve over time depending on the past forecasting performance of the two types of agents. Similar dynamic expectation formation was found by Branch 2004 [5] in the Michigan Survey of inflationary expectations. He finds evidence that agents switch predictor use as the relative mean squared errors change. Agents' predictor choices respond negatively to increases in relative mean square error.

Our weighting algorithm follows similar pattern: if the forecasting error of LS is smaller than that of RE, agents put more weight on LS. On average the weight on LS ω counts how many times the forecasts of LS were better than the forecasts of RE: ω is the average of an indicator function I^{LS} which takes the value 1 whenever the LS forecast was closer to the actual outcome than the RE forecast, 0 when RE is better. The mathematical interpretation in the limit is clear-cut: ω gives the *probability that forecasts of LS are better*

[7]The basic model of asset pricing under risk neutrality takes the same form, with p_t interpreted as the price of stock, m_t as its dividend. $\lambda = \frac{1}{1+r}$ is the one period discount factor, r the rate of return on the riskless asset.

than forecasts of RE.

$$\begin{aligned}
\omega_t &= \frac{\sum_{i=1}^{t-1} I_i^{LS} + k}{t+k} \quad k \in \mathbb{R}^+ \\
I_i^{LS} &= \begin{cases} 1 & \text{if } \left|E_{i-1}^{LS} p_i - p_i\right| < \left|E_{i-1}^{RE} p_i - p_i\right| \\ 0 & \text{else} \end{cases}
\end{aligned}$$

Given $E_0^{LS} p_1$ and $E_0^{RE} p_1$. Or equivalently in a recursive formulation[8]:

$$\omega_t = \omega_{t-1} + \frac{1}{t+k}\left(I_{t-1}^{OLS} - \omega_{t-1}\right) \tag{4}$$

In our simulations we set the first period weight on LS to 1, i.e. the whole population is assumed to follow least squares learning. Then in subsequent periods if the forecast of RE is better, the weight on LS decreases. We smooth initial fluctuations of ω, which can be easily done by setting k to be a big number.

The novelty in this approach is that there is a dynamic predictor selection. Heterogeneity is not exogenously fixed but evolves over time based on the performance of the least squares learner and the rational agent.

One way to think about this model is that aggregate expectations are formed by a representative agent, who does not have any sophisticated model at hand to form his expectations. He simply observes the forecasts of two agents, and weights them according to how good they forecasted in the past. Agents for example could simply read the official inflation forecasts of the central bank and of the ministry of finance, and then decide which to believe more. If the ministry notoriously underestimated inflation compared to the central bank, clearly agents will believe in its new forecast less.

An other way to interpret this model is as a heterogeneous agents model, where the type that is more successful in his forecasts will be more dominant in the economy. An example for this could be a population of firms using a pricing algorithm. If one pricing algorithm performs worse than the other, more and more firms will switch to the other one.

[8]Note that the weighting algorithm would be the same if we used squared forecasting error: $I_i^{LS} = 1$ if $\left(E_{t-1}^{LS} p_t - p_t\right)^2 < \left(E_{t-1}^{RE} p_t - p_t\right)^2$.

1.1 Least Squares Learning

We consider two learning algorithms to model the first type of agents. One simply takes averages of past inflation, the other observes the money supply, m and runs a regression in the correct Minimum State Variable (MSV) specification. We will examine how our results change depending on the specification of the learning algorithm.

The first learning algorithm observes only past price levels, thus the best he can do is to run regression on a constant (henceforth LS_1).

$$LS_1 \qquad E_t p_{t+1}^{LS_1} = a_t \quad a_t = \frac{\sum_{i=1}^{t-1} p_i}{t-1}$$

Or in a recursive formulation

$$a_t = a_{t-1} + \frac{1}{t}(p_{t-1} - a_{t-1}) \tag{5}$$

The second learning algorithm is 'more clever', also observes the money supply m and runs regression in the MSV form (henceforth LS_2). Note that MSV solution of (1) (2) is $\mathrm{E}_t p_{t+1} = \frac{\varrho}{1-\lambda\rho} m_t$.

$$LS_2 \qquad E_t p_{t+1}^{LS_2} = \beta_t m_t \quad \beta_t = \frac{\sum_{i=1}^{t-1} p_i m_{i-1}}{\sum_{i=1}^{t-1} m_{i-1}^2}$$

So LS_2 hypothesizes that the price level is leaded by last period's money supply. He runs a regression of price on lagged money supply, and then makes his forecast of next period's price level with his latest estimated coefficient and the current period's money supply. The recursive formulation of the regression coefficient is

$$\beta_t = \beta_{t-1} + \frac{1}{t-1}\frac{1}{R_{t-1}} m_{t-2}\left(p_{t-1} - m_{t-2}\beta_{t-1}\right) \tag{6a}$$

$$R_t = R_{t-1} + \frac{1}{t}\left(m_{t-1}^2 - R_{t-1}\right) \tag{6b}$$

where R_t is the moment matrix[9].

[9] $R_t = \frac{\sum_{i=1}^{t} m_i m_i'}{t}$

1.2 Rational Expert

Our next modelling choice concerns the second type of agents, the rational experts. Again we consider two specifications. Both know the model with its parameters, the stochastic process of m and observe the LS forecasts. The difference is that one of them observes the evolution of the weights while the other does not.

The first expert, RE_1 has a misperception and thinks the the whole population follows least squares learning. Since for the simulations we set the initial weight on LS to 1, RE_1 is rational in the first period, we could say he is 'one-step rational'. In the subsequent periods RE_1 does not observe changes in the weights and will get further away from rationality. Still, in the initial periods he will be close to rationality since weights on LS are not let to decrease quickly.

In every period he forms his expectations by using E^{LS} in (1) and using the stochastic process of m, (2) (further details follow):

$$RE_1 \qquad E_t^{RE_1} p_{t+1} = \lambda E_t^{LS} p_{t+2} + \varrho m_t$$

The second expert RE_2 always observes the weights. At time t he can calculate the true conditional expectation of p_{t+1} (further details follow later) conditional on his information set.

$$RE_2 \qquad E_t^{RE_2} p_{t+1} = E[p_{t+1}|\Omega_t] \qquad \Omega_t = \{E_t^{LS}; \omega_t; m_t; \lambda; \varrho\}$$

RE_1 can be interpreted as an agent who investigated a lot in discovering the true parameters of the economy but considers himself too small to influence expectations. One could hypothesize for example that a forecasting agency must have the capacity to investigate thoroughly the underlying economy, but does not believe or does not know exactly how his forecasts are influencing aggregate expectations, to what extent agents "believe" his forecast. RE_2 on the other hand believes his forecasts are followed by agents, think for example of the central bank. Furthermore RE_2 can even investigate the 'credibility' of his forecasts and use it to improve his future forecasts.

Summing it up, our model consists of the underlying economy, equations (1) and (2) and the aggregate inflationary expectation, equation (3). This latter in turn is based on the forecasts of two types of agents: least squares learners and rational agents weighted according to their average past performance, equation (4). We will examine convergence properties conditional on how sophisticated are these forecasts: whether the rational agent observes the weight on the two types of agents or not, and the least squares learner has correctly specified regression (6) or misspecified regression (5). We examine whether the introduction of a well informed expert implies some different model behavior compared to the standard least squares learning case. Does it converge to a different equilibrium? What are the equilibrium weights on least squares learners and rational agents? Is its rate of convergence different from that of LS learning?

2 Equilibrium under Least Squares Learning with Rational Experts

In this section we examine convergence of our economy. As a benchmark we compare convergence of LS_1 and LS_2. Then adding rational agents there are several interesting questions to be examined. Firstly, in our paper weights on the two types of agents evolve over time. Establishing convergence results for the weights answers whether least squares learning 'survives' in the presence of rational agents. Further, we examine whether the presence of rational agents modifies the equilibrium and the conditions of convergence. We also examine how different specifications for the rational agents affect our results.

2.1 Benchmark: Convergence Under Least Squares Learning

The two learning algorithms can potentially learn very different equilibria depending on their specification. LS_1 is not conditioning on the state variable m, so the equilibrium he can achieve is a restricted perceptions equilibrium. Since LS_1 is only taking averages of past inflation , the best he can achieve is to find the true unconditional expectation of the price level. LS_2 on the other hand is running a regression in the MSV form thus has a 'chance' to learn the MSV rational expectations equilibrium.

When the economy is populated by LS_2 learners, the aggregate expectation in (1) is given by their Percived Law of Motion (henceforth PLM), $\widetilde{E}_t p_{t+1} = E_t^{LS_2} p_{t+1} = \beta_t m_t$. Using this we can obtain the Actual Law of Motion of the price level (henceforth ALM):

$$p_t = (\lambda\beta_t + 1)(\varrho m_{t-1} + \varepsilon_t)$$

Heuristically an equilibrium β is one where agents perceptions about the future price level become justified by the actual outcome, i.e. the possible resting point of β is the fixed point of a mapping from the PLM to the ALM. It can be easily shown that β_t converges to the rational expectations solution $\beta_f = \frac{\varrho}{1-\lambda\varrho}$ (henceforth β^{RE}) given $\lambda\varrho < 0$. (See Appendix) Further, the equilibrium is the MSV rational expectations equilibrium $p_t = \frac{\varrho}{1-\lambda\varrho}m_t$.

When the economy is populated with LS_1 agents aggregate expectations are given by their PLM $E_t^{LS_1}p_{t+1} = a_t$. Using this expectation in (1) gives the ALM

$$p_t = \lambda a + m_t$$

In the appendix we show that a_t converges to $a_f = 0$ if $\lambda < 1$. Thus, the equilibrium is $p_t = m_t$, which is different from the case of LS_2.

We have seen that the equilibrium is intrinsically different depending on the learning algorithm agents follow. When agents condition on m, under the economy converges to the MSV rational expectations equilibrium, given $\lambda\varrho < 1$. When agents only take averages of past inflation, the equilibrium will be a so called restricted perceptions equilibrium, given $\lambda < 1$. When the conditions $\lambda\varrho < 1$ or $\lambda < 1$ respectively do not hold learning does not converge and the economy does not settle down in an equilibrium.

2.2 Least Squares Learning in the MSV form -LS_2- in the Presence of a Rational Expert

Let us first examine the case when LS learning is specified in the MSV form. In the previous section we have shown that under certain parameter values learning converges to the MSV rational equilibrium. This section examines whether LS_2 survives in the presence of a rational agent and whether conditions for convergence are modified by the presence of an expert, RE_1 and RE_2 in turn.

Let us first consider a population with LS_2 and RE_1 agents. Recall that this expert is 'one-step rational', he knows initially all agents follow LS_2, knows the underlying economy so at the beginning is indeed able to calculate rational expectations. However he does not observe further evolution of the weights and mistakenly thinks the whole population continues doing learning. It follows from the self referential nature of the model that there is an interesting feedback from the rational agent's forecast to his forecasting performance. If RE_1 has more accurate forecasts he will be more credible, will have a higher weight. On the other hand through the higher weight he will have a higher influence on aggregate expectations, which effect he is not considering in forming his expectations. Thus, his higher weight in turn will worsen his forecasting performance. His equilibrium weight will be a fixed point where these two opposite effects cancel each other.

Expert RE_1 forms his expectations using his knowledge about the underlying economy equations (1)-(2) and the forecast of the least squares learner. His PLM is: $p_t = \lambda E_t p_{t+1}^{LS} + m_t = (\lambda\beta_t + 1)m_t$. Then with his knowledge about m following AR(1) process he makes his next period‘s forecast[10] as

$$E_t p_{t+1}^{RE_1} = E_t[(\lambda\beta_t + 1)m_{t+1}] = (\lambda\beta_t + 1)\rho m_t \tag{7}$$

Next consider expert RE_2 who has a better information set than the previous one, observes the weights denoted to him and the least squares learner. One can find his forecasting function with guess and verify[11]. In general the guess should be $p_t = a(\omega_t)E_t^{LS}p_{t+1} + b(\omega_t)m_t$. When the LS expectations are in the form $E_t^{LS}p_{t+1} = \beta_t m_t$ it is equivalent to guess $p_t = b(\omega_t)m_t$[12]. It can be easily verified that RE_2's Perceived Law of Motion and forecast for the next period is [13]:

$$p_t = \frac{1+\lambda\omega_t\beta_t}{1-\lambda\varrho\,(1-\omega_t)}m_t \qquad \text{PLM of } RE_2 \tag{8}$$

$$E_t^{RE_2}p_{t+1} = \varrho\frac{1+\lambda\omega_t\beta_t}{1-\lambda\varrho\,(1-\omega_t)}m_t \tag{9}$$

For the equilibrium with the two specifications we have to determine the Actual Law of Motion of the economy. For this one has to use the expectations $E_t^{LS_2}p_{t+1}$ and (8) or (7) respectively and the underlying economy (1)-(3).

[10]Notice that for simplicity we suppose the rational agent observes the forecast of the learner but does not consider the updating algorithm of β. In other words the rational agent is myopic, thinks that the estimated β of LS will remain unchanged from now on. From the construction of RE_1 it follows that he also considers $\omega = 1$ to be unchanged.

[11]The same method is used, with the same guess in Nunes 2004, where he models learning with rational expectations in a New Keynesian framework, with fixed weights.

[12]Footnote 10 applies here too: for simplicity we suppose the rational agent is myopic, thinks that the estimated β and ω of LS will remain unchanged from now on. Since RE_2 observes the weights he can calculate p_t and the current forecasting error of LS. Then using (6) in principle he could calculate how β will be updated next period. This would lead to a complicated solution for the rational expectation. Since solving this is not in the focus of the paper, we apply the simplifying assumption that rational agents are myopic.

[13]Or equivalently the PLM of RE_2 is $p_t = \frac{\lambda\omega_t}{1-\lambda\varrho\,(1-\omega_t)}E_t^{LS}p_{t+1} + \frac{1}{1-\lambda\varrho\,(1-\omega_t)}m_t$

It can be easily shown that for the economy with LS_2 and RE_1 The Actual Law of Motion for the price level is

$$p_t = [\lambda\,(\omega_t\beta_t + (1-\omega_t)(\lambda\beta_t + 1)\varrho) + 1]\, m_t$$

The ALM for the population with LS_2 and RE_2 is identical to the PLM of this expert: equation (8). This is not of a surprise, since RE_2 knows everything to calculate the current price level, in this sense he is fully rational.

Notice that event though RE_2 can calculate the current price level, the next periods price level contains a shock ε that he does not foresee. Further as we mentioned in the footnotes of this section experts do not consider the updating algorithm of β and ω, so they cannot calculate the true conditional expectation of next periods p as a truly fully rational agent would do.

In equilibrium agents perceptions should become true, i.e. the equilibrium is a fixed point of a mapping from the PLM to the ALM. Since agents are forecasting next periods price level we have to compare their PLM for one period ahead and the ALM of p next period as a function of current variables[14].

Proposition 1. *Let the economy* (1)-(4) *be populated with two types of agents* LS_2 *and* RE_1*.* $[\beta_t, \omega_t]$ *converge to* $[\beta^{RE}, \frac{1}{2}]$ *if* $\lambda \in (-\frac{2}{\varrho}, \frac{1}{\varrho})$*, where* $\beta^{RE} = \frac{\varrho}{1-\lambda\varrho}$ *is the* β *corresponding to the rational expectations equilibrium.*

Proposition 2. *Let the economy* (1)-(4) *be populated with two types of agents* LS_2 *and* RE_2*.* $[\beta_t, \omega_t]$ *converge to* $[\beta^{RE}, \frac{1}{2}]$ *if* $\lambda < \frac{1}{\varrho}$*, where* $\beta^{RE} = \frac{\varrho}{1-\lambda\varrho}$ *is the* β *corresponding to the rational expectations equilibrium.*

Proof. See the Appendix. □

The above propositions state a surprising result: even though LS_2 has much less information than a rational agent he will not die out. Moreover in the long run LS_2 will have the same weight as a rational agent. Recall that the equilibrium weight corresponds to the probability that the forecast of LS is closer to the actual outcome than the forecasts of RE. This means that least squares learning in the MSV form not only survives competition with a rational agent, but in the limit it has a 50% chance to have a better forecast than the rational agent.[15]

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[14]For details see the Appendix.

[15]Simulations with perturbing the initial weights confirm that equilibrium weights do not depend on the initial weights.

An intuition can be gained by the following simple example. For the sake of easier algebra let us take the expert to be RE_1 and choose a positive λ. For easier exposition we drop the time subsript and use the following notation: $\varrho(1+\lambda\beta) \equiv \beta^{RE_1}$ and $[\lambda\varrho\,(\omega\beta+(1-\omega)(\lambda\beta+1)\varrho)+\varrho] \equiv \beta^{ALM}$. Then the forecast of RE_1 for p_{t+1} is $\beta^{RE_1}m_t$ and the ALM of $p_{t+1} = \beta^{ALM}m_t + \frac{1}{\varrho}\beta^{ALM}\varepsilon_{t+1}$.

Let us consider first an economy without shocks. Then is is easy to show that out of equilibrium, when $\beta \neq \frac{\varrho}{1-\lambda\varrho}$ forecasts of RE_1 would always perform better forecasts than LS_2. Mathematically, for $\beta < \frac{\varrho}{1-\lambda\varrho}$ we have $\beta < \beta^{RE_1} < \beta^{ALM}$ and for $\beta > \frac{\varrho}{1-\lambda\varrho}$ we have $\beta > \beta^{RE_1} > \beta^{ALM}$. So β^{RE_1} is always closer to β^{ALM} then β^{LS_2}. In such an economy LS_2 would die out.

With shocks, even if β^{RE_1} is always closer to β^{ALM} then β^{LS_2}, a large enough ε might push the price level closer to the forecast of LS_2. As β gets closer to the rational equilibrium, the forecasts of the learner and the expert also get closer to each other and even a smaller shock might make the LS_2 forecast better. In other words the learner has higher and higher probability to forecast better than the expert as the economy converges closer and closer to the equilibrium. In the limit the expectations of the learner and the expert are negligibly close to each other and will have the same probability to perform better than the other[16]

The above propositions also state a striking difference between an economy with RE_1 and RE_2: compared to the benchmark with only LS_2 agents, RE_2 does not alter the condition for convergence while RE_1 does. There is a range of coefficient values $\lambda < -\frac{2}{\varrho}$ for which LS_2 converges without an expert, but with expert RE_1 does not converge. In other words the presence of a rational agent not having correct perceptions might introduce instability.

2.3 Least Squares Learning About a Constant -LS_1- in the Presence of a Rational Expert

Now let us turn to LS_1 when least squares learners are running a misspecified regression, with PLM $E_t^{LS_1}p_{t+1} = a_t$. Before we have shown that for $\lambda < 1$ learning converges to 0, the true unconditional expectation of the price under rational expectations. This section, similarly to the previous one, considers

[16]We implicitly assumed that the distribution of the shock around β^{ALM} is not varying in time. This is not true since in the ALM $p_{t+1} = \beta_{t+1}^{ALM}m_t + \frac{1}{\varrho}\beta_{t+1}^{ALM}\varepsilon_{t+1}$ so ε is multiplied by a coefficient varying in time. However this does not change the intuition behind.

convergence in the presence of an expert.

As before, the first expert RE_1 does not observe the weights, he is making forecasts with the initial weight $\omega = 1$ and with the knowledge of m following an AR(1) process. His forecast for the next period is:

$$E_t^{RE_1} p_{t+1} = \lambda a_t + \varrho m_t \tag{10}$$

Expectations of the second expert, RE_2 can be solved with the same guess as in the previous section. We guess $p_t = c(\omega_t)E_t^{LS} p_{t+1} + d(\omega_t)m_t$, which implies $E_t^{RE_2} p_{t+1} = c(\omega_t)a_t + d(\omega_t)\varrho m_t$. It can be verified that $c = \frac{\lambda\omega_t}{1-\lambda(1-\omega_t)}$ and $d = \frac{1}{1-\lambda\varrho(1-\omega_t)}$. Thus

$$E_t^{RE_2} p_{t+1} = \frac{\lambda\omega_t}{1-\lambda(1-\omega_t)} a_t + \frac{1}{1-\lambda\varrho(1-\omega_t)} \varrho m_t \tag{11}$$

To determine the Actual Law of Motion of the economy one has to use the expectations of the learner, $E_t^{LS_1} p_{t+1}$ and of the expert, (8) or (7) respectively and the underlying economy (1)-(3).

The Actual Law of Motion with expert RE_1 is:

$$p_t = \left(\lambda\omega_t + \lambda^2 (1-\omega_t)\right) a_t + (\lambda\varrho (1-\omega_t) + 1) m_t \qquad \text{ALM with } RE_1$$

The Actual Law of Motion with the second expert is equal to the PLM of RE_2.

$$p_t = \frac{\lambda\omega_t}{1-\lambda(1-\omega_t)} a_t + \frac{1}{1-\lambda\varrho(1-\omega_t)} m_t \quad \text{ALM with } RE_2$$

Proposition 3. *Let the economy* (1)-(4) *be populated with two types of agents* LS_1 *and* RE_1. $[a_t, \omega_t]$ *converge to a unique fixed point* $[0, \omega_f]$, $\omega_f \in (0, 0.5]$ *if* $\lambda\omega + \lambda^2(1-\omega) < 1$.

Proposition 4. *Let the economy* (1)-(4) *be populated with two types of agents* LS_2 *and* RE_2. $[\beta_t, \omega_t]$ *converge to a unique fixed point* $[0, \omega_f]$, $\omega_f \in (0, 0.5]$ *if* $\lambda < 1$.

Lemma 1. $\frac{\partial\omega_f}{\partial\varrho} < 0$ *with* LS_1 *and* RE_1.*Also* $\frac{\partial\omega_f}{\partial\varrho} < 0$ *with* LS_1 *and* RE_2.

Proof. See the Appendix. □

The most surprising result in these theorems is that least squares learning LS_1 does not 'dye out' in the presence of a rational expert. In other words the weight on LS does not collapse to 0.

This means that learning survives forecasting competition with a rational agent even if it has a misspecified form. LS_1 eventually learns the true unconditional expectation of p under rational expectations and with positive probability its forecasts are closer to the actual outcome than forecasts of the rational agent[17]. Notice also that the equilibrium weight on LS_1 is never bigger than the equilibrium weight on the expert, the misspecified learning algorithm naturally performs worse than a rational agent[18].

Lemma (1) states an interesting result, that we did not have with learning in the MSV form. The equilibrium weights depend on the persistence of autoregressive process. The higher is ϱ the smaller is the weight on LS_1. The intuition behind is simple: the more persistent is m, the longer time it takes for a shock on m to die out, and the bigger mistake it is not using data on m. So, a more persistent stochastic process for m makes the forecasts of LS_1 become worse compared to RE who conditions on m. When $\varrho = 0$, m is a random noise so conditioning on it does not help forecasting more then just taking averages of p: the weights of LS_1 will be equal to the weights of the rational expert[19].

In equilibrium LS_1 learns the same restricted perceptions equilibrium as without the presence of a rational agent: $a_f = 0$. However, because heterogeneity is always an equilibrium outcome (given conditions for convergence hold) the equilibrium itself is different with an expert than without an expert.

The ALM under LS_1 and RE_1 is: $p_t = (\lambda\varrho(1-\omega_f)+1)m_t$. Under LS_1 and RE_1 the equilibrium is: $p_t = \frac{1}{1-\lambda\varrho(1-\omega_f)}m_t$.Thus the equilibrium also depends on the specification of the rational expert.

Notice also that the form of equilibrium depends on the equilibrium weight. Interestingly, with an under-parametrized learning, LS_1 we can still get close to the MSV rational expectation solution when a correctly specified

[17]This would not be the case for an agent forecasting simply a constant. In this case its weight would converge to 0 unless its forecast is sufficiently close to 0.

[18]For LS_1 finding the exact value of equilibrium weight cannot be done analytically. In this case finding ω_f is a complicated fixed point problem of a function inside the normal cumulative distribution function. (See Appendix.) Since the latter does not have a closed form solution we cannot give exact analytical results. However it is easy to get numerical solutions.

[19]$\omega_f = 0.5$ if $\varrho = 0$ follows directly from the Appendix.

expert, RE_2, is present in the economy. With RE_2 when ϱ is sufficiently high so that ω_f is close to zero, the equilibrium will be close to the rational equilibrium. In other words, when the persistence of m is very high, LS_1 forecasters will perform so badly that experts will dominate the equilibrium. When the expert is correctly specified, RE_2, the economy can get close to the MSV rational expectation solution. On the other when the expert has a misspecified model, RE_1, and learning has an under-parametrized regression the equilibrium will be different from the MSV solution.

Finally, a result similar to the previous section is that a rational agent with a misspecified model might introduce instability. The presence of RE_1 again decreases the parameter set for which convergence to the equilibrium applies. Convergence of LS_1 with or without RE_2 has the same condition.

3 Speed of Convergence of Least Squares Learning in the Presence of a Rational Expert

The previous section showed that the presence of a rational agent does not alter where least squares learning converges, as long as conditions for convergence are satisfied. LS_2 converges to the MSV rational expectations solution, while LS_1 converges to a Restricted Perceptions Equilibrium, learns the true unconditional expectation of the price level in the rational equilibrium. Throughout this section we will suppose convergence is not an issue: we set $\lambda \in (0, 1)$ thus conditions for convergence are always met. We will examine whether the presence of a rational agent can speed up convergence of least squares learning. We derive analytical speed of convergence results, then in the next section we provide finite sample rate of convergence results by Monte Carlo simulations.

Our main motivation for examining speed of convergence is that least squares learning can converge very slowly (see Marcet and Sargent 1995 [20]). We are interested in how the presence of a rational agent can speed up convergence to the equilibrium; and how important is the information set of the rational agent.

Slow convergence can indeed be a problem for several reasons. When learning converges slowly, expectations will be out of the rational equilibrium most of the time. Then making decisions assuming rational expectations would be erroneous even if learning had been present for a long time and we know that in the long run it converges to the rational expectations equilibrium. Slow convergence also implies that the asymptotic distribution for test statistics can be very different when agents follow LS learning compared to when agents have rational expectations. When speed of convergence to rational expectations is very slow, the confidence intervals will be larger than the confidence intervals from classical econometrics. This means that an econometrician who derives confidence intervals assuming agents follow LS learning will reject the null hypothesis less frequently, than an econometrician who derives asymptotic distribution for test statistics supposing rational expectations.

We examine speed of convergence applying the theorem of Benveniste, Méivier and Priouret 1990 [4] (theorem 3, page 110).

Let θ_t be the vector of parameter estimates, X_t the state vector, and γ_t the deterministic sequence of gains. The function $\mathcal{Q}$ expresses the way in which the estimates of θ are updated from period $t-1$ to t. In our case $\gamma_t = \frac{1}{t}$. With LS_1 $\theta_t = [a_t, \omega_t, R_t]$ and with LS_2 $\theta_t = [\beta_t, \omega_t, R_t]$. The state vector is $X_t = [m_{t-2}, \varepsilon_{t-1}]$. $\mathcal{Q}$ is the updating term in the recursive formulation in equations θ (5),(6),(4) respectively.

From the recursive formulation

$$\theta_t = \theta_{t-1} + \gamma_t \mathcal{Q}(t, \theta_{t-1}, X_t)$$

define

$$h(\theta) = E[\mathcal{Q}(\theta, X_t)]$$

for fixed θ. Let θ_f be such that $h(\theta_f) = 0$. The theorem of Benveniste et al. concludes that if the Jacobian of $h(\theta)$ evaluated at θ_f has all eigenvalues less than $-\frac{1}{2}$ in real part then

$$t^{0.5}(\theta_t - \theta_f) \quad \xrightarrow{\mathcal{D}} \quad N(0, P)$$

where the matrix P satisfies

$$\left[\frac{1}{2}h_\theta(\theta_f)\right] P + P \left[\frac{1}{2}h_\theta(\theta_f)\right]' + E\mathcal{Q}(\theta, X_t)\mathcal{Q}(\theta, X_t)' = 0$$

Thus if the above conditions are met we have root t convergence to θ_f. Also, for higher eigenvalues of the Jacobian convergence is slower in the sense that the asymptotic variance-covariance is higher. So even when there is root-t convergence, higher eigenvalues of $h_\theta(\theta_f)$ imply slower convergence.

3.1 Benchmark: if there is only least squares learning

To understand better the theorem of Benveniste et al. and also as a point of comparison, let us first examine speed of convergence under LS_2. From the recursive formulation for the coefficients, equation (6), it is easy to show:

$$h\begin{pmatrix}\beta \\ R\end{pmatrix} = \begin{bmatrix} \frac{1}{R}M_m[(\lambda\beta+1)\varrho-\beta] \\ M_m - R \end{bmatrix}$$

Where $E[m_{t-2}m_{t-2}] = \frac{\sigma_\varepsilon^2}{1-\varrho^2} = M_m$. The Jacobian of $h(\theta)$ is

$$\frac{\partial h(\theta)}{\partial \theta} = \begin{bmatrix} \frac{1}{R}M_m(\lambda\varrho - 1) & \frac{-1}{R^2}M_m[(\lambda\beta+1)\varrho-\beta] \\ 0 & -1 \end{bmatrix}$$

We have to evaluate this at the fixed point $[\beta\ R] = [\beta^{RE}\ M_m]$. The two eigenvalues are: -1 and $\lambda\varrho - 1$. For $\sqrt{t}$ convergence these have to be smaller than $-\frac{1}{2}$, so under LS_2 we need $\lambda\varrho < \frac{1}{2}$. [20]

Similar calculations for LS_1 yield: for $\sqrt{t}$ convergence under LS_1 we need $\lambda < \frac{1}{2}$.

One immediate difference between LS_2 and LS_2 is that the persistence of the autoregressive process influences speed of convergence in the latter but not in the former. When least squares learning does not condition on m the stochastic properties of m will have no effect on how quickly learning gets to the equilibrium. On the other hand when least squares learning conditions on m the more persistent is m the less information can be gained from variations in m (the PLM will be closer to the ALM) and the slower learning will find the equilibrium.

3.2 Speed of convergence of learning in MSV form, LS_2

This section examines whether the present of a rational expert increases speed of convergence of least squares learning in the MSV form. Above we derived, for root-t convergence of LS_2 $\lambda\varrho < \frac{1}{2}$ has to hold, if the presence of an expert speeds up convergence root-t convergence has to apply for a wider range of coefficients.

Proposition 5. *Let the economy* (1)-(4) *be populated with two types of agents* LS_2 *and* RE_1. *Then* LS_2 *has* $\sqrt{t}$ *convergence if* $\lambda\varrho < \frac{2}{3}$.

[20] For further discussion see Marcet et al. [20].

Proposition 6. *Let the economy* (1)-(4) *be populated with two types of agents* LS_2 *and* RE_2*. Then* LS_2 *has* $\sqrt{t}$ *convergence if* $\frac{1}{2}\lambda\varrho + \frac{1}{2}\lambda^2\varrho^2 < \frac{1}{2}$.

Proof. See the Appendix. □

Corollary 1. *Speed of convergence of least squares learning in the MSV form is decreasing in* ϱ*. This is also true in the presence of a rational agent - either* RE_1 *or* RE_2*.*

The importance of these propositions is that the presence of a rational agent $\sqrt{t}$ convergence applies for a wider range of coefficients when least squares learners have access to the forecast of a well informed expert. Moreover $\sqrt{t}$ convergence applies to a wider set of coefficients when the rational expert has a better information set.

The corollary states a finding also derived by Marcet et. al [20] for LS_2 and found by Timmerman 1996 [30] the more persistent is the stochastic process the slower least squares learning will converge to the true parameters.

3.3 Speed of convergence for learning about a constant -$LS1$- in the presence of a rational expert

This chapter derives conditions of root-t convergence for LS_1 in the presence of a rational expert. Recall that without an expert the condition for $\sqrt{t}$ convergence is $\lambda < 1$. Without a rational expert the nature of the stochastic process of m does not alter speed of convergence, since LS_1 is not conditioning on m.

Proposition 7. *Let the economy* (1)-(4) *be populated with two types of agents* LS_1 *and* RE_2*. Then* LS_1 *has* $\sqrt{t}$ *convergence if* $\lambda(1 + \omega_f) < 1$.

Proposition 8. *Let the economy* (1)-(4) *be populated with two types of agents* LS_1 *and* RE_1*. Then* LS_1 *has* $\sqrt{t}$ *convergence if* $\lambda\omega_f + \lambda^2(1 - \omega_f) < \frac{1}{2}$.

Lemma 2. *Least squares learning about a constant,* LS_1*, in the presence of a rational agent has higher speed of convergence when* ϱ *is higher.*

Proof. See the Appendix. □

Results of Proposition 7 and 8 are parallel to results for LS_2. The presence of a rational agent increases speed of convergence. Also the better the rational agent's information set the more it increases speed of convergence of LS_1.

The effect of ϱ, the persistence of the autoregressive process, now has an opposite effect as in the case of LS_2. Lemma 2 shows that speed of convergence is now increasing in ϱ, while with LS_2 a higher persistence of m caused slower rate of convergence.

The intuition behind this result is simple. Recall, that when the economy is populated with LS_1 agents, its speed of convergence does not depend on the autoregressive parameter of m, since this agent does not condition its forecast on m. However, in the presence of a rational agent the equilibrium weight of LS_1 does depend on ϱ (Lemma 1) and this in turn effects speed of convergence.

The more persistent the money process is the bigger mistake is not conditioning on it: a higher ϱ leads to a lower equilibrium weight on LS_1. A lower weight in turn means that the forecasts of LS_1 influence less the actual outcome. The actual price level is mainly determined by expectations of the rational agent who conditions on m, thus the least squares learner will discover he is 'doing the wrong thing' sooner. In other words his Perceived Law of Motion will be very different from the Actual Law of Motion and this leads to a higher speed of convergence.

Analytical results of this section show that the presence of a rational agent does increase speed of convergence of least squares learning. This finding is independent of the learning algorithm, both LS_1 and LS_2 converges faster. The better the information set of the rational agent the more its presence speeds up convergence of learning.

Learning in the MSV form has slower speed of convergence the more persistent the autoregressive process m is. This finding is parallel to the findings of Marcet and Sargent [20] and Timmermann 1996 [30]. Marcet and Sargent [20] derived analytical speed of convergence results for the same model withLS_2. Timmermann examined a stock pricing equation with learning where the exogenous process was an autoregressive dividend process. By simulations he found that a more persistent dividend process results slower convergence of learning.

Our results show that this result is only true if learning conditions on the exogenous process. When learning does not condition on m its speed of convergence does not depend on the persistence of m. Moreover with endogenous weights in the presence of a rational agent a misspecified learning algorithm can even converge faster with a more persistent m.

4 Simulations

In this section we provide some numerical results on finite sample rate of convergence with Monte Carlo simulations.

Following the methodology of Marcet and Sargent 1995 [20] the Monte Carlo simulations are based on the assumption that there is a δ for which

$$t^{\delta}(\beta_t - \beta_f) \xrightarrow{\mathcal{D}} F \tag{12}$$

for some non-degenerate well-defined distribution F with mean zero and variance σ_F^2. Then $t^{\overline{\delta}}(\beta_t - \beta_f) \to 0$ for $\overline{\delta} < \delta$ and we will call δ the rate of convergence of β_t.

Intuitively, the faster β_t is converging to β_f, the higher t^{δ} has to be to ensure that the product on the left hand side of 12 converges to a proper distribution.

Equation 12 implies that $E[t^{\delta}(\beta_t - \beta_f)]^2 \to \sigma_F^2$ as $t \to \infty$. Therefore

$$\frac{E[t^{\delta}(\beta_t - \beta_f)]^2}{E[(tk)^{\delta}(\beta_{tk} - \beta_f)]^2} \to 1$$

which implies that

$$\frac{(E\beta_t - \beta_f)^2}{E(\beta_{tk} - \beta_f)^2} \to k^{2\delta} \qquad \text{as} \qquad t \to \infty$$

This justifies using

$$\delta = \frac{1}{\log(k)} \log \left[\frac{E(\beta_t - \beta_f)^2}{E(\beta_{tk} - \beta_f)^2)} \right]^{1/2}$$

for large t in finite samples, as an approximation for the rate of convergence.

Given t and k the expectations involved can be approximated by Monte Carlo integration: calculating a large number of independent realizations of length t and tk and calculating the mean square difference from β_f.

For the simulations we calculated the rates of convergence with 1000 independent realizations. For initial conditions we set the least squares β and R equal to their limiting point $\beta_0 = \beta_f$ (or $a = a_f$), $R_0 = \frac{1}{1-\varrho^2}$. The initial weight on the least squares learner was set equal to 1, k in equation 4 was set equal to 0.9. Several seeds of the random number generator were tried, rates of convergence were maximum 0.03 from each other for the long sample, and 0.06 in the short sample.

In Table 1 analytical results on speed of convergence are summarized. The first row indicates conditions for $\sqrt{t}$ convergence. Since in the simulations we use $\varrho = 0.9$ in the second line of Table 1 we summarize the cutoff values of λ for root-t convergence with $varrho = 0.9$. For example when the economy is populated with LS_2 and RE_2 agents and $\varrho = 0.9$ λ has to be smaller than 0.74 for root-t convergence.

Table 1: **Conditions for $\sqrt{t}$ convergence**

	LS_1	$LS_1 + RE_1$	$LS_1 + RE_2$	LS_2	$LS_2 + RE_1$	$LS_2 + RE_2$
$\sqrt{t}$ conv	$\lambda < 0.5$	$\lambda\omega + \lambda^2(1-\omega) < 1$	$\lambda(1-\omega) < 1$	$\lambda\varrho < 0.5$	$\frac{\lambda\varrho+\lambda^2\varrho^2}{2} < 0.5$	$\lambda\varrho < \frac{2}{3}$
$\varrho = 0.9$	0.5	0.67	0.8	0.55	0.61	0.74

Table 2: **Speed of convergence of ls learning in the MSV form (in short and long sample)** $\varrho = 0.9$

	LS_1		With RE_1		With RE_2	
λ	t=200 to 800	t=2000 to 10,000	t=200 to 800	t=2000 to 10,000	t=200 to 800	t=2000 to 10,000
0.1000	0.5623	0.5130	0.5614	0.5116	0.5614	0.5115
0.1450	0.5635	0.5152	0.5627	0.5126	0.5627	0.5125
0.1900	0.5638	0.5184	0.5640	0.5139	0.5641	0.5137
0.2350	0.5621	0.5223	0.5650	0.5157	0.5652	0.5151
0.2800	0.5574	0.5265	0.5656	0.5179	0.5661	0.5170
0.3250	0.5488	0.5296	0.5654	0.5206	0.5665	0.5190
0.3700	0.5354	0.5299	0.5637	0.5236	0.5659	0.5211
0.4150	0.5171	0.5252	0.5601	0.5263	0.5642	0.5232
0.4600	0.4941	0.5138	0.5536	0.5284	0.5611	0.5245
0.5050	0.4670	0.4954	0.5435	0.5283	0.5563	0.5248
0.5556	0.4328	0.4669	0.5273	0.5233	0.5477	0.5227
0.6400	0.3701	0.4066	0.4871	0.4990	0.5253	0.5095
0.6850	0.3350	0.3704	0.4580	0.4761	0.5084	0.4939
0.7300	0.2995	0.3328	0.4238	0.4472	0.4886	0.4720
0.7750	0.2637	0.2943	0.3862	0.4126	0.4635	0.4438
0.8200	0.2278	0.2553	0.3436	0.3728	0.4334	0.4082
0.8650	0.1920	0.2159	0.2966	0.3257	0.3972	0.3673
0.9100	0.1563	0.1764	0.2481	0.2695	0.3615	0.3215
0.9950	0.0891	0.1016	0.1453	0.1627	0.2409	0.2217

Tables 2 and 3 summarize simulation results for least squares learning in the MSV form, LS_2 and least squares learning about a constant LS_1. The estimated δs are presented, for $\sqrt{t}$ convergence these have to be higher than 0.5.

Table 3: **Speed of convergence of ls learning about a constant (in short and long sample)** $\varrho = 0.9$

	LS_1		With RE_1		With RE_2	
λ	t=200 to 800	t=2000 to 10,000	t=200 to 800	t=2000 to 10,000	t=200 to 800	t=2000 to 10,000
0.1000	0.4661	0.5023	0.4714	0.5043	0.4718	0.5045
0.1450	0.4610	0.5004	0.4699	0.5039	0.4706 0.5042	
0.1900	0.4546	0.4978	0.4679	0.5033	0.4693	0.5038
0.2350	0.4467	0.4942	0.4654	0.5025	0.4677	0.5034
0.2800	0.4370	0.4892	0.4621	0.5014	0.4658	0.5028
0.3250	0.4254	0.4823	0.4579	0.4998	0.4635	0.5022
0.3700	0.4116	0.4728	0.4527	0.4977	0.4608	0.5012
0.4150	0.3955	0.4601	0.4459	0.4948	0.4575	0.5001
0.4600	0.3769	0.4436	0.4373	0.4905	0.4535	0.4986
0.5050	0.3557	0.4229	0.4265	0.4844	0.4483	0.4966
0.5556	0.3288	0.3943	0.4110	0.4740	0.4414	0.4935
0.6400	0.2769	0.3350	0.3743	0.4435	0.4241	0.4841
0.6850	0.2459	0.2985	0.3476	0.4173	0.4111	0.4754
0.7300	0.2128	0.2594	0.3151	0.3823	0.3934	0.4619
0.7750	0.1778	0.2184	0.2763	0.3378	0.3688	0.4408
0.8200	0.1412	0.1760	0.2306	0.2840	0.3355	0.4071
0.8650	0.1032	0.1327	0.1779	0.2220	0.2894	0.3549
0.9100	0.0639	0.0888	0.1184	0.1530	0.2224	0.2749
0.9950	-0.0128	0.0047	-0.0113	0.0086	-0.0016	0.0195

The simulations in general coincide with the analytical results. Rate of convergence of least squares learning is higher with expert advice. Also convergence is faster the better is the information set of the rational agent. Simulations also show that a rational agent speeds up convergence more when learning is misspecified compared to when learning is correctly specified.

From the simulation results one can observe that convergence is very low for high values of λ. This confirms the theorem of Benveniste, Méivier and Priouret 1990 [4], the higher the eigenvalues of $h_\theta(\theta_f)$ the slower is the rate of convergence. Also rate of convergence can be slower than $\frac{1}{2}$, when the conditions in the Benveniste et al. theorem do not hold.

Conclusion

Modelling expectations remains a controversy: rational expectations is criticized for acquiring too much knowledge from agents, adaptive learning which is an alternative is criticized for being ad hoc and for having slow convergence to the rational equilibrium.

Admittedly, the choice of a learning algorithm is necessarily ad hoc. However, our paper shows that if agents forecast with a learning algorithm and have access to forecasts of a rational agent, they will not rush to abandon their ad hoc learning rule. For this, we developed a self-referential model with two types of agents, learners and rationals, where weights on them evolved according to their past forecasting performance.

Our paper showed, that the coexistence of learners and rationals can be rationalized in an equilibrium framework. Surprisingly, learning survives forecasting competition with a rational agent even if it is misspecified.

Our results coincide with recent surveys of inflation expectations, which find that expectations are well represented as being a weighted average of forward-looking and backward-looking expectations.

We believe our results strengthen the case for modelling expectations as a mixture of adaptive and forward looking expectations. Especially, since it is well documented that modelling expectations this way improves empirical performance of standard models.

Our second main result is that the presence of a rational agent 'helps' learning algorithms to converge faster. The more rationality the rational agent possesses the bigger is the set of coefficients for $\sqrt{t}$ convergence. This result is also confirmed by finite sample by simulations.

This result might further strengthen the case for using learning models enriched with rational agents to model expectations, since criticisms against learning for its slow convergence might be weakened.

An other way to think about our results is that we show how the presence of a rational expert, for example a central bank, might influence an economy where agents try to form their expectations with past data, behaving as econometricians. A practical example might be an economy with learners and a central bank having rational expectations. If the central bank would hesitate between making its forecasts public or not, our paper suggests the answer is not straightforward.

An economy with rational experts and learners performs better than an economy with only least squares learners, in the sense that learners are converging faster to the equilibrium. Since agents will be out of equilibrium for a shorter time, welfare costs of departing from the equilibrium are reduced.

However, we have seen that the presence of a rational agent might even introduce instability to the system when the rational agent uses a misspecified model. In this sense making the forecasts of the rational expert accessible for learners might even worsen the situation.

Our advice for a central bank considering making its forecasts public or not would then be: yes, if they are sure they do not induce instability of expectations. Making central bank forecasts public would help agents to learn faster. However, if the central bank has a misspecified model, publishing central bank forecasts might even make learning impossible.

A Appendix

A.1 Convergence Proofs

Throughout our convergence proofs we build on technical results of stochastic approximation. So first let us briefly describe this method. Let us consider the following stochastic recursive algorithm (SRA)

$$\theta_t = \theta_{t-1} + \gamma_t \mathcal{Q}(\theta_{t-1}, X_t)$$

where θ_t is a vector of parameter estimates, X_t is the state vector, and γ_t is a deterministic sequence of gains. The function $\mathcal{Q}$ expresses the way in which the estimates of θ are updated from period $t-1$ to t.

If $\mathcal{Q}$ and X_t satisfies some technical assumptions (See Evans and Honkapohja 2001 Chapter 6 [12][21])

the stochastic approximation approach associates an ordinary differential equation (ODE) with the SRA form

$$\frac{d\theta}{d\tau} = h(\theta(\tau))$$

where $h(\theta)$ is obtained as

$$h(\theta) = \lim_{t\to\infty} E\mathcal{Q}(\theta, X_t(\theta))$$

for a fixed θ provided this limit exists. τ denotes "notional" or "artificial" time.

According to the results established by stochastic approximation theory, if this ODE has an equilibrium point[22] θ_f which is locally asymptotically stable[23], then θ_f is a possible point of convergence of the algorithm. If θ_f is not a locally stable equilibrium point of the ODE, then θ_f is not a possible point of convergence of the SRA, i.e. $Pr(\theta \to \theta_f) = 0$.

Claim 1. *Let β_t evolve according to* (6) *in the model* (1)-(2). *Then β_t converges to $\beta^{RE} = \frac{\varrho}{1-\lambda\varrho}$ given $\lambda\varrho < 1$*

Proof. Since the root of the autoregressive part of the process generating m lies outside the unit circle we can use stochastic approximation. It can be easily verified

[21] Ljung's Theorems 4 and 2. [17]

[22] θ_f is an equilibrium point if $h(\theta_f) = 0$

[23] θ_f is locally stable if all eigenvalues of the derivative matrix (Jacobian) $Dh(\theta_f)$ have negative real parts.

that $h(\theta)$ is

$$\frac{d\beta}{d\tau} = \frac{1}{R} M_m[(\lambda\beta + 1)\varrho - \beta] \tag{13}$$

$$\frac{dR}{d\tau} = M_m - R \tag{14}$$

where $M_m = \frac{\sigma_\varepsilon^2}{1-\varrho^2}$ is the limiting variance of m. The fixed point is $\theta_f = [\frac{\varrho}{1-\lambda\varrho}\ M_m]$. Which is a stable convergence point if $\lambda\varrho < 1$.[24] □

Claim 2. *Let a_t evolve according to (5) in the model (1)-(2). Then a_t converges to 0 given $\lambda < 1$*

Proof. It can be easily verified that $h(\theta)$ is

$$\frac{da}{d\tau} = \lambda a - a \tag{15}$$

The fixed point is $a = 0$ which is stable if $\lambda < 1$. □

Proof of Proposition 2 is omitted, it goes along the same lines as proof of proposition 1.

Proof of Proposition 1

Proof. Both β and ω and R evolve over time and we have to analyze their joint dynamical system. We will use stochastic approximation.

Our model in SRA form:

$$\begin{aligned} \beta_t &= \beta_{t-1} + \frac{1}{t}\frac{1}{R_{t-1}} m_{t-2}\{[\ \lambda[\omega_{t-1}\beta_{t-1} + (1-\omega_{t-1})(\lambda\beta_{t-1}+1)\varrho] + 1\ ]\varrho m_{t-2} - \beta_{t-1}m_{t-2} + \\ &\quad [\lambda[\omega_{t-1}\beta_{t-1} + (1-\omega_{t-1})(\lambda\beta_{t-1}+1)\varrho] + 1]\varepsilon_{t-1}\} \\ R_t &= R_{t-1} + \frac{1}{t}\left(m_{t-1}^2 - R_{t-1}\right) \\ \omega_t &= \omega_{t-1} + \frac{1}{t}\left(I_{t-1}^{LS} - \omega_{t-1}\right) \end{aligned}$$

[24] In the case of OLS it is sufficient to examine E-stability. Marcet and Sargent 1989 [19] proved that convergence of the learning scheme in this case is related to E-stability.

In our case $\theta_{t-1} = [\beta_{t-1}, \omega_{t-1}, R_{t-1}]$. γ_t is $\frac{1}{t}$ and $X_t = [m_{t-2}, \varepsilon_{t-1}]$[25].[26] Using $E[m_{t-2}m_{t-2}] = \frac{\sigma_\varepsilon^2}{1-\varrho^2} = M_m$ and $E[m_{t-2}\varepsilon_{t-1}] = 0$ the associated system of ordinary differential equations, $\frac{d\theta}{d\tau} = h(\theta)$, is:

$$\frac{\partial \beta}{\partial \tau} = \frac{1}{R} M_m [\lambda\varrho(\omega\beta + (1-\omega)(\lambda\beta+1)\varrho) + \varrho - \beta] \quad (16)$$

$$\frac{\partial R}{\partial \tau} = M_m - R \quad (17)$$

$$\frac{\partial \omega}{\partial \tau} = E[I^{LS}(\theta)] - \omega \quad (18)$$

Where $E[I^{LS}(\theta)]$ is a complicated function to be determined bellow. First let us recall that a realization of I^{LS} at time t is

$$I_t^{LS} = \begin{cases} 1 & \text{if } \left|E_{t-1}^{LS} p_t - p_t\right| \leq \left|E_{t-1}^{RE} p_t - p_t\right| \\ 0 & \text{else} \end{cases}$$

Where $E_{t-1}^{LS} p_t = \beta_{t-1} m_{t-1}$, $E_{t-1}^{RE} = (\lambda\beta_{t-1}+1)\varrho m_{t-1}$, $p_t = [\lambda[\omega_t\beta_t + (1-\omega_t)(\lambda\beta_t + 1)\varrho] + 1]\varrho m_{t-1} + [\lambda[\omega_t\beta_t + (1-\omega_t)(\lambda\beta_t + 1)\varrho] + 1]\varepsilon_t)$. Thus I_t^{LS} is a function $I_t^{LS} = I(\theta_t, \theta_{t-1}, m_{t-1}, \varepsilon_t)$. To use stochastic approximation we have to fix θ and evaluate $\lim_{t\to\infty} E(I(\theta, \theta, m_{t-1}, \varepsilon_t))$. Since $I^{LS} = 1$ when the LS forecast was better than the RE forecast, and 0 otherwise, the expected value of the function $I(\cdot)$ gives us on average how many percentage of all cases the LS is expected to be better than RE. In other words $E(I(\cdot))$ gives what is the probability that LS is better than RE.

$$\lim_{t\to\infty} E(I(\theta, \theta, m_{t-1}, \varepsilon)) = \lim_{t\to\infty} Pr(|A(\theta)m_{t-1} - C(\theta)\varepsilon_t| \leq |B(\theta)m_{t-1} - C(\theta)\varepsilon_t|)$$

Where $A(\theta) = \beta - [\lambda[\omega\beta + (1-\omega)(\lambda\beta+1)\varrho] + 1]\varrho$, $B(\theta) = (\lambda\beta+1)\varrho - [\lambda[\omega\beta + (1-\omega)(\lambda\beta+1)\varrho] + 1]\varrho$. $C(\theta) = [\lambda[\omega_{t-1}\beta_{t-1} + (1-\omega_{t-1})(\lambda\beta_{t-1}+1)\varrho] + 1]$. For simplicity

[25] In equation (16) and (18) the original tracking parameters are $\gamma_t = \frac{1}{t-1}$ and $\gamma_t = \frac{1}{t+k}$. It is easy to verify that rewriting them in the standard SRA form with $\gamma_t = \frac{1}{t}$ results a second-order complementary term which does not affect the associated ODE.

[26] The regularity conditions A1, A.2, A.3 are easy to verify (we use notation of Evans and Honkapohja 2001 [12]). To verify assumption A.2. we have to restrict R_{t-1} to be bounded away from 0 thus the domain for $\theta_t = (\beta_t, R_t, \omega_t)$ is $(\mathbb{R}, D = (R_L, \infty), [0, 1])$ where $R_L \geq \epsilon > 0$. This is a natural restriction since R_t is the sample moment of m. Assumption A.3. is satisfied since $\mathcal{Q}(\theta, x)$ is twice continuously differentiable with bounded second derivatives. Assumptions B.1. and B.2. are satisfied since m is an AR(1) process and ε is white noise.

we will slightly abuse notation and do not denote explicitly the dependence of A, B and C on θ.

We can simplify using $Pr(|x| \leq |y|) = Pr(x^2 \leq y^2)$. Then depending on how we fixed θ we can further simplify:

$$\begin{aligned}
\lim_{t\to\infty} \quad & Pr(C(B-A)m_{t-1}\varepsilon_t \leq \frac{B^2-A^2}{2}m_{t-1}^2) = \\
\lim_{t\to\infty}\{ \quad & Pr(m_{t-1}\varepsilon_t \leq \frac{B+A}{2C}m_{t-1}^2 \ , \ B-A>0, \ C>0) + Pr(m_{t-1}\varepsilon_t \leq \frac{B+A}{2C}m_{t-1}^2 \ , \ B-A<0, \ C<0) \\
& Pr(m_{t-1}\varepsilon_t \geq \frac{B+A}{2C}m_{t-1}^2 \ , \ B-A<0, \ C>0) + Pr(m_{t-1}\varepsilon_t \geq \frac{B+A}{2C}m_{t-1}^2 \ , \ B-A>0, \ C<0)\}
\end{aligned}$$

We have to distinguish 4 cases taking care of how the sign of $B-A$ and C changes depending on how we fixed θ. Notice that $B-A>0$ if $\beta < \frac{\varrho}{1-\lambda\varrho} = \beta^{REE}$, so we have $B-A>0$ if we have fixed β bellow the one corresponding to the rational expectations equilibrium.

Next we rewrite the problem as $\lim_{t\to\infty} E(Pr_{t-1}(\ \cdot \ ))$. So we first take the probability conditional on time $t-1$ for a given history of the state variables up to $t-1$. The functional form of this probability will depend on whether the realization of m_{t-1} is positive or negative. Next we take the unconditional expectation and the limit.

First let us consider the cases for $\forall \ \theta$s.t. $B-A>0, C>0$ or $B-A<0, C<0$. Then conditional on the sign of the realization of m_{t-1} we can write. $\lim_{t\to\infty} E\{ \ (Pr_{t-1}(\varepsilon_t \leq \frac{B+A}{2C}m_{t-1} \ , \ m_{t-1}>0) + Pr_{t-1}(\varepsilon_t > \frac{B+A}{2C}m_{t-1} \ , \ m_{t-1}<0)) \ \} = \lim_{t\to\infty} E\{ \ [\Phi_\varepsilon(\frac{B+A}{2C}m_{t-1} \ , \ m_{t-1}>0) + \Phi_\varepsilon(-\frac{B+A}{2C}m_{t-1} \ , \ m_{t-1}<0)] \ \} = \lim_{t\to\infty} E \ \Phi_\varepsilon(\frac{B+A}{2C}|m_{t-1}|)$. Where Φ_ε is the cumulative distribution function of ε, recall that ε follows normal distribution $N(0,\sigma_\varepsilon^2)$. $|m_{t-1}|$ is the absolute value of m_{t-1}. Taking the limit results: $\int \Phi_\varepsilon(\frac{B+A}{2C}|m|)d\phi_m$, where ϕ_m is the limiting normal distribution of m [27].

For $\forall \ \theta$s.t. $B-A<0, C>0$ or $B-A>0, C<0$ similar derivation yield to $\int \Phi_\varepsilon(-\frac{B+A}{2C}|m|)d\phi_m$.

Summarizing our calculations:

$$E[I^{LS}(\theta)] = \begin{cases} \int \Phi_\varepsilon(\ \frac{B+A}{2C}|m| \ )d\phi_m, & \forall\theta \text{ s.t. } B-A>0, C>0 \text{ or } B-A<0, C<0 \\ \int \Phi_\varepsilon(\ -\frac{B+A}{2C}|m| \ )d\phi_m, & \forall\theta \text{ s.t. } B-A<0, C>0 \text{ or } B-A>0, C<0 \end{cases} \quad (19)$$

Next let us examine the equilibrium points of the system of differential equations (16)-(18). From equation (17) the possible convergence point for $R_t = \frac{\sum_{i=1}^t m_i}{t}^2$ is $R_f = M_m$, which is the variance of the limiting distribution of m.

[27] $\Phi_\varepsilon(\frac{B+A}{2C}|m_{t-1}|)$ is bounded and measurable thus we can interchange the limit and the integral (Lebesgue's Dominated Convergence Theorem).

Then from equation (16) the equilibrium β is the rational expectations equilibrium

$$\beta_f = \frac{\lambda(1-\omega)\varrho^2+\varrho}{1-\omega\lambda\varrho-(1-\omega)\lambda^2\varrho^2} = \frac{(\lambda(1-\omega)\varrho+1)\varrho}{(\lambda(1-\omega)\varrho+1)(1-\lambda\varrho)} = \frac{\varrho}{1-\lambda\varrho} \tag{20}$$

To find ω_f we need to calculate the fixed point of (19) using $\beta_f = \frac{\varrho}{1-\lambda\varrho}$, $R_f = M_m$. For this notice that $B\begin{pmatrix}\beta_f\\R_f\\\omega\end{pmatrix} - A\begin{pmatrix}\beta_f\\R_f\\\omega\end{pmatrix} = 0$. Also $B\begin{pmatrix}\beta_f\\R_f\\\omega\end{pmatrix} + A\begin{pmatrix}\beta_f\\R_f\\\omega\end{pmatrix} = 0$, $C\begin{pmatrix}\beta_f\\R_f\\\omega\end{pmatrix} \neq 0$ thus

$$E\left[I^{LS}\begin{pmatrix}\beta_f\\R_f\\\omega\end{pmatrix}\right] = \int \Phi_\varepsilon(0|m|)\ d\phi_m = \frac{1}{2} \tag{21}$$

Our result is that the fixed point of ω is $\omega_f = \frac{1}{2}$. Thus the equilibrium of the system of differential equations (16)-(18) is

$$\theta_f = [\beta_f, R_f, \omega_f] = \left[\frac{\varrho}{1-\lambda\varrho}, M_m, \frac{1}{2}\right]$$

where M_m is the variance of the limiting distribution of m.

It remains to show that θ_f is a locally stable equilibrium point of (16)-(18). For this we have to find the eigenvalues of the derivative matrix (Jacobian) evaluated at θ_f, $Dh(\theta)|_{\theta_f}$. θ_f is locally stable if all eigenvalues have negative real parts.

$$Dh(\theta) = \begin{bmatrix} \frac{M_m}{R}[\lambda\varrho\omega+\lambda^2\varrho^2(1-\omega)-1] & \frac{-M_m}{R^2}[\lambda\varrho(\omega\beta+(1-\omega)(\lambda\beta+1)\varrho)+\varrho-\beta] & \frac{M_m}{R}[\lambda\varrho\beta-\lambda\varrho^2(\lambda\beta+1)] \\ 0 & -1 & 0 \\ \frac{\partial E[I^{LS}(\theta)]}{\partial\beta} & 0 & \frac{\partial E[I^{LS}(\theta)]}{\partial\omega}-1 \end{bmatrix}$$

Evaluating $Dh(\theta)$ at θ_f gives

$$Dh(\theta)|_{\theta_f} = \begin{bmatrix} \lambda\varrho\frac{1}{2}+\lambda^2\varrho^2\frac{1}{2}-1 & 0 & 0 \\ 0 & -1 & 0 \\ \left.\frac{\partial E[I^{LS}(\theta)]}{\partial\beta}\right|_{\theta_f} & 0 & \left.\frac{\partial E[I^{LS}(\theta)]}{\partial\omega}\right|_{\theta_f}-1 \end{bmatrix}$$

This is a lower triangular matrix so the eigenvalues are the diagonal elements. Let us evaluate the third eigenvalue $\left.\frac{\partial E[I^{LS}(\theta)]}{\partial \omega}\right|_{\theta_f} - 1$.

Let us consider $\forall\ \theta$ s.t. $B(\theta) - A(\theta) > 0, C > 0$ or $B(\theta) - A(\theta) < 0, C < 0$. (For $\forall\ \theta$ s.t. $B(\theta) - A(\theta) > 0, C < 0$ or $B(\theta) - A(\theta) < 0, C > 0$ calculations go similarly.) Then from (19)

$$\left.\frac{\partial E[I^{LS}(\theta)]}{\partial \omega}\right|_{\theta_f} = \int \frac{\partial}{\partial \omega}\Phi_\varepsilon\left(\frac{B+A}{2C}|m|\right) d\phi_m\Big|_{\theta_f} = \int \frac{\partial}{\partial \omega}\frac{B+A}{2C}|m|\phi_\varepsilon\left(\frac{B+A}{2C}|m|\right) d\phi_m\Big|_{\theta_f}$$

Where ϕ_ε is the distribution function of ε[28]. $\left.\frac{\partial}{\partial \omega}\frac{(B+A)}{2C}\right|_{\theta_f} = 0$ also $A(\theta_f) + B(\theta_f) = 0$ and $C(\theta_f) \neq 0$. Thus

$$\int \frac{\partial}{\partial \omega}\frac{B+A}{2C}|m|\phi_\varepsilon\left(\frac{B+A}{2C}m\right) d\phi_m\Big|_{\theta_f} = 0\int \phi_\varepsilon\left(0\right) d\phi_m\Big|_{\theta_f} = 0$$

The Jacobian at θ_f is

$$Dh(\theta)|_{\theta_f} = \begin{bmatrix} \lambda\varrho\frac{1}{2} + \lambda^2\varrho^2\frac{1}{2} - 1 & 0 & 0 \\ 0 & -1 & 0 \\ \left.\frac{\partial E[I^{LS}(\theta)]}{\partial \beta}\right|_{\theta_f} & 0 & -1 \end{bmatrix}$$

θ_f is locally stable if all eigenvalues are negative. Clearly the last two eigenvalues are negative. The first one is negative if $\lambda\varrho\frac{1}{2} + \lambda^2\varrho^2\frac{1}{2} - 1 < 0$. This condition holds if $\lambda \in (-\frac{2}{\varrho}, \frac{1}{\varrho})$, where $\varrho \in [0, 1)$ by assumption.

□

Proof of Proposition 3 is omitted, it goes along the same lines as proof of proposition 4.

Proof of Proposition 4

Proof. The proof goes similarly to proof of Proposition (1). We have to find the fixed point of $\theta = [a\ \omega]$. The associated system of ordinary differential equations,

[28] We could interchange the derivative and the integral since $F_\varepsilon\left(\frac{B+A}{2C}\right)$ is a continuous function $\mathbb{R} \to \mathbb{R}$ (thus measurable).

$\frac{d\theta}{d\tau} = h(\theta)$, is:

$$\frac{\partial a}{\partial \tau} = \frac{\lambda\omega}{1-\lambda(1-\omega)}a - a \tag{22}$$

$$\frac{\partial \omega}{\partial \tau} = E[I^{LS}(\theta)] - \omega \tag{23}$$

Where $E[I^{LS}(\theta)] = \lim_{t\to\infty} E\{\ (Pr_{t-1}(\varepsilon_t \leq \frac{A^2a^2+B^2m_{t-1}^2+2ABam_{t-1}}{2ACa+2BCm_{t-1}}\ ,\ 2ACa + 2BCm_{t-1} < 0) + Pr_{t-1}(\varepsilon_t > \frac{A^2a^2+B^2m_{t-1}^2+2ABam_{t-1}}{2ACa+2BCm_{t-1}}\ ,\ 2ACa + 2BCm_{t-1} > 0)\ \}$ With $A = \left(1 - \frac{\lambda\omega}{1-\lambda(1-\omega)}\right)$, $B = \left(-\frac{\varrho}{1-\lambda(1-\omega)}\right)$ and $C = \left(-\frac{1}{1-\lambda(1-\omega)}\right)$.

From (22) it is easy to see that the equilibrium a is $a_f = 0$. Then using $\theta_f = [a_f\ \omega_f] \in [0\ [0,1]]$ in (23) some algebra leads to

$$\omega_f = \int \Phi_\varepsilon\left(\ \frac{-\varrho}{2}|m|\ \right) d\phi_m \tag{24}$$

The uniqueness of $\theta_f = [0\ \omega_f]$ follows from the fact that the left hand side of (24) is independent of ω. $\omega_f \in (0, 0.5]$ follows from (24) and $\varrho \in [0, 1)$.

$\theta_f = [0\ \omega_f]$ is a locally stable equilibrium point of (22)-(23) if the eigenvalues of the derivative matrix (Jacobian) evaluated at θ_f, $Dh(\theta)|_{\theta_f}$ are negative in modulus.

The Jacobian at θ_f is

$$Dh(\theta)|_{\theta_f} = \begin{bmatrix} \frac{\lambda\omega_f}{1-\lambda(1-\omega_f)} - 1 & 0 \\ \left.\frac{\partial E[I^{LS}(\theta)]}{\partial a}\right|_{\theta_f} & \left.\frac{\partial E[I^{LS}(\theta)]}{\partial \omega}\right|_{\theta_f} - 1 \end{bmatrix}$$

This is a lower triangular matrix, the eigenvalues are the diagonal elements. Through some lengthy but straightforward calculations it can be shown that $\left.\frac{\partial E[I^{LS}(\theta)]}{\partial \omega}\right|_{\theta_f} < 0$. Thus the second eigenvalue is negative. The first eigenvalue is negative if $\lambda < 1$. □

Proof of Lemma 1

Proof. Follows from equation (24). □

A.2 Speed of convergence proofs

Proof of Proposition 6

Proof. **Speed of convergence of LS_2 with RE_1**
In the proof of Proposition 1 we have already derived:

$$Dh(\theta)|_{\theta_f} = \begin{bmatrix} \lambda\varrho\frac{1}{2} + \lambda^2\varrho^2\frac{1}{2} - 1 & 0 & 0 \\ 0 & -1 & 0 \\ \frac{\partial E[I^{LS}(\theta)]}{\partial \beta}\Big|_{\theta_f} & 0 & -1 \end{bmatrix}$$

This is a lower triangular matrix, the eigenvalues are the diagonal elements. Clearly the las two eigenvalues are smaller than $-\frac{1}{2}$. The first one is smaller than $-\frac{1}{2}$ if $\frac{1}{2}\lambda\varrho + \frac{1}{2}\lambda^2\varrho^2 < \frac{1}{2}$. □

Proof of Proposition 5

Proof. **Speed of convergence of LS_2 with RE_2**

$$Dh(\theta)|_{\theta_f} = \begin{bmatrix} \frac{\lambda\varrho}{2-\lambda\varrho} - 1 & 0 & 0 \\ 0 & -1 & 0 \\ \frac{\partial E[I^{LS}(\theta)]}{\partial \beta}\Big|_{\theta_f} & 0 & -1 \end{bmatrix}$$

This is a lower triangular matrix, the eigenvalues are the diagonal elements. For root-t convergence all eigenvalues have to be smaller than $-\frac{1}{2}$, which holds if $\lambda\varrho < \frac{2}{3}$. □

Proof of Proposition 7

Proof. **Speed of convergence of LS_1 with RE_2**
Using derivations in the proof of Proposition 4:

$$Dh(\theta)|_{\theta_f} = \begin{bmatrix} \frac{\lambda\omega_f}{1-\lambda(1-\omega_f)} - 1 & 0 \\ \frac{\partial E[I^{LS}(\theta)]}{\partial a}\Big|_{\theta_f} & \frac{\partial E[I^{LS}(\theta)]}{\partial \omega}\Big|_{\theta_f} - 1 \end{bmatrix}$$

With $\frac{\partial E[I^{LS}(\theta)]}{\partial \omega}\Big|_{\theta_f} < 0$.

This is a lower triangular matrix, the eigenvalues are the diagonal elements. For root-t convergence the eigenvalues have to be smaller than $-\frac{1}{2}$, which holds if

$$\lambda(1 + \omega_f) < 1$$

□

Proof of Proposition 8

Proof. **Speed of convergence of LS_1 with RE_1**

$$\frac{\partial h(\theta)}{\partial \theta}\bigg|_{\theta_f} = \begin{bmatrix} \lambda\omega_f + \lambda^2(1-\omega_f) - 1 & 0 \\ \frac{\partial E[I^{LS}(\theta)]}{\partial a}\Big|_{\theta_f} & \frac{\partial E[I^{LS}(\theta)]}{\partial \omega}\Big|_{\theta_f} - 1 \end{bmatrix}$$

With $\frac{\partial E[I^{LS}(\theta)]}{\partial \omega}\Big|_{\theta_f} < 0$.

Since this is a lower triangular matrix the eigenvalues are the diagonal elements. Eigenvalues are smaller than 0.5 if $\lambda\omega_f + \lambda^2(1-\omega_f) < \frac{1}{2}$. □

Proof of Lemma 2

Proof. In the case of LS_1 with RE_2 the proof follows from Proposition (7) and Lemma 1.

In the case of LS_1 with RE_1 follows similarly from Proposition (8) and and Lemma 1 with the following calculations.

From Proposition (8) the condition for $\sqrt{t}$ convergence is $\lambda\omega_f + \lambda^2(1-\omega_f) < \frac{1}{2}$. For a given ω_f this is a quadratic equation in λ with solutions $\lambda \in (\underline{\lambda}, \overline{\lambda})$ where $\underline{\lambda}, \overline{\lambda} \in \mathbb{R}, \underline{\lambda} \leq \overline{\lambda}$.

We have to show $\frac{\partial \overline{\lambda}}{\partial \varrho} = \frac{\partial \overline{\lambda}}{\partial \omega_f}\frac{\partial \omega_f}{\partial \varrho} > 0$.

Lemma 1 already showed $\frac{\partial \omega_f}{\partial \varrho} < 0$. It remains to show $\frac{\partial \overline{\lambda}}{\partial \omega_f} < 0$. This on the other hand can be easily shown by taking the derivative of $\overline{\lambda} = \frac{-\omega_f + \sqrt{\omega_f^2 - 2\omega_f + 2}}{2(1-\omega_f)}$ with respect to ω_f and using $\omega_f \in (0, 0.5]$ from Proposition 3. □

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